PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Geomagnetic Dynamos

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GEOMAGNETIC DYNAMOS

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CONTENTS

The 'dynamo theory' ascribes the origin of the earth's magnetic field to the dynamo action of motions in the conducting fluid of the earth's core. This paper supports the theory by proving rigorously that it is possible to postulate a pattern of motions in a sphere filled with conducting fluid in such a way that the arrangement acts as a dynamo producing a magnetic field extending outside the conductor. The equations of motion of the fluid are ignored.

The proof is given for a model consisting of two eddies in the earth's core, and does no more than demonstrate that motions in a sphere filled with conducting fluid can act as a steady dynamo. It is certainly not suggested that the motions in the earth's core are so simple.

There is nothing pathological about the relative orientations of the angular velocity vectors of the two eddies which lead to dynamo action; in fact about half of the possible relative orientations work.

1. Introduction

Larmor (1919) suggested that the earth's magnetic field might have its origin in the dynamo action of motions in the conducting fluid of the earth's core. A considerable literature has grown up around Larmor's suggestion (reviews have been published by Elsasser 1950, 1955, 1956*a*, *b*; Cowling 1953, 1957; Inglis 1955; Jacobs 1956; Runcorn 1956; Hide 1956), but so far it has not been shown that motions in a sphere filled with conducting fluid can act as a dynamo producing a magnetic field which extends outside the conductor. The purpose of this paper is the limited one of showing that dynamo action *is* possible if the velocity pattern is postulated without regard to the equations of fluid motion. The principal difference from previous work with the same object is that the proof given here is rigorous, whereas the previous work (Takeuchi & Shimazu 1953; Bullard & Gellmann 1954) used series expansions which had to be cut off without justification.

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We shall consider a model consisting of a sphere filled with conducting matter.† Within the sphere, we imagine two smaller spheres such that the distance between their centres exceeds the sum of their radii. The conducting matter filling up the interior of each small sphere rotates as a rigid body at a constant angular velocity about a fixed axis. The conductor outside the small spheres is stationary. The two small spheres will be referred to as A and B. The model acts as a dynamo in which A rotates in a magnetic field arising from induction in B, the magnetic field applied to B being due to induction in A. In other words, the two spheres feed one another.

This model is adequate to demonstrate the possibility of a dynamo due to motions in a conducting sphere, but no more. It is extremely unlikely that the motions in the earth's core are so simple. Indeed, there is indirect evidence (Bullard, Freedman, Gellman & Nixon 1950) for an angular velocity gradient, possibly due to the effect of the conservation of angular momentum on convective motions. Such a gradient might be important in a terrestrial dynamo (Bullard & Gellmann 1954; Bullard *et al.* 1950; see however § 11), but our model makes no use of it. If one wants a physical picture, then one can look on the two spheres in our model as convection cells, or eddies, but this point of view will not be developed.

All we shall do to show that our model is not irrelevant from the point of view of physics is to prove that the relative orientations of the axes of rotation of the two spheres required for dynamo action are in no way improbable (in fact about half the possible relative orientations work), and that when the dimensions are of the order of the earth's core, the velocities which the model requires are larger by about an order of magnitude than the velocity at which the non-dipole part of the earth's magnetic field drifts westward at the surface of the core (Bullard *et al.* 1950).

The plan of the paper is as follows. An outline of the analysis is given in § 2. In §§ 3–6 it is shown that the question of whether our model can act as a dynamo can be reduced to the problem of the existence of the solution of a system of linear simultaneous equations with an infinite number of variables. The proof that a solution *can* exist is given in §§ 7 and 8, and the proof that the corresponding magnetic field extends outside the conductor in § 9. The velocities required for dynamo action are estimated in § 10. A general discussion is given in § 11, and the conclusions are stated in § 12.

Much of the paper may be omitted on a first reading. The essential points of the argument will be covered by reading § 2, § 3 down to equation (3·6), and §§ 6, 8, 11 and 12. The reader will have no difficulty in finding the definitions of the symbols used there in the missed sections.

2. Conditions for the existence of a dynamo

We shall consider the arrangement shown in figure 1. It consists of a rigid sphere of conductivity σ and radius M surrounded by insulating material. Embedded in this sphere are two small rigid spheres A and B, which also have conductivity σ, and which are in perfect electric contact with their surroundings over the whole of their surfaces; their radius is a.

The large sphere is stationary in the laboratory system of reference (which we suppose to be inertial). The small spheres A and B rotate at constant angular velocities $\boldsymbol{\omega}_A$ and $\boldsymbol{\omega}_B$ about axes passing through their respective centres. We shall assume that $|\boldsymbol{\omega}_A| = |\boldsymbol{\omega}_B| = \omega$,

† A greatly shortened and simplified account of the argument of this paper will be published in *Annales de Géophysique*.

but this restriction is of no importance because we shall see that only the product $|\boldsymbol{\omega}_A||\boldsymbol{\omega}_B|$ plays an essential role.

The centres of A and B are at $\mathbf{R}_A$ and $\mathbf{R}_B$, respectively, the origin being taken at O, the centre of the large sphere. The position of any point is denoted either with respect to the origin by $\mathbf{r}$, or with respect to $\mathbf{R}_A$ or $\mathbf{R}_B$ by $\mathbf{r}_A$ or $\mathbf{r}_B$. The vector pointing from $\mathbf{R}_A$ to $\mathbf{R}_B$ is denoted by $\mathbf{R}$; i.e. $\mathbf{R} = -\mathbf{R}_A + \mathbf{R}_B = +\mathbf{r}_A - \mathbf{r}_B$.

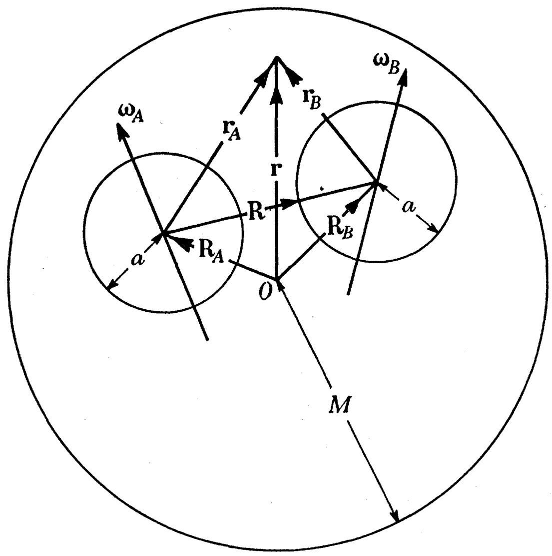

FIGURE 1. For $r < M$, σ = constant $\neq 0$; for $r > M$, $\sigma = 0$.

We shall suppose that everything is time independent. The field equations are

$$\operatorname{curl} \mathbf{H} = \mathbf{J}, \tag{2·1}$$

$$\operatorname{curl} \mathbf{E} = 0, \tag{2·2}$$

$$\operatorname{div} \mathbf{H} = 0, \tag{2·3}$$

$$\operatorname{div} \mathbf{E} = qc^2. \tag{2·4}$$

The symbols $\mathbf{H}$ and $\mathbf{E}$ are to denote the magnetic and electric field vectors as they appear in the laboratory system. $\mathbf{J}$ is the current density, and q the density of electric charge. We are using rationalized e.m.u. c is the velocity of light *in vacuo*. The permeability and dielectric constant have been taken to be equal to unity.

$\mathbf{J}$ is given by

$$\mathbf{J} = 0 \quad \text{in} \quad r > M, \tag{2·5a}$$

$$= \sigma \mathbf{E} \quad \text{in} \quad r < M, \text{ but outside } A \text{ and } B, \tag{2·5b}$$

and

$$= \sigma(\mathbf{E} + \mathbf{v} \wedge \mathbf{H}) \quad \text{inside } A \text{ and } B, \tag{2·5c}$$

where $\mathbf{v}$ is the local velocity of matter as seen in the laboratory. The expression (2·5c) for the current density inside the rotating spheres has been computed by the use of Lorentz transformations, neglecting a convection current density $q\mathbf{v}$ and terms of order v^2/c^2.

At the surfaces of A and B, the vector $\mathbf{H}$ and the tangential components of the vector $\mathbf{E}$ are continuous. These boundary conditions follow from (2·1) to (2·5) in the usual way. The same conditions hold at the surface $r = M$, and in addition the normal component of $\mathbf{J}$ has to vanish on the inside if there is to be no surface charge density varying with time.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

The problem to be discussed is whether it is possible to dispose of $\boldsymbol{\omega}_A$, $\boldsymbol{\omega}_B$, a, $\mathbf{R}$, $\mathbf{R}_A$, $\mathbf{R}_B$ and M in such a way that equations (2·1) to (2·5) and the boundary conditions can be satisfied by *non-vanishing* fields $\mathbf{E}$ and $\mathbf{H}$ which vanish as $r \to \infty$, and for which $\mathbf{H} \neq 0$ at some points in $r > M$. Arrangements which permit of such solutions will be called dynamos.

We shall now formulate the conditions for our model to act as a dynamo. Let us suppose that spheres A and B have applied to them electromagnetic fields $(\mathscr{A})$ and $(\mathscr{B})$, respectively; these symbols denote both the electric and magnetic components. The fields $(\mathscr{A})$ and $(\mathscr{B})$ will eventually be generated by the arrangement in figure 1 itself; for the moment we just take them to be given; they are to satisfy the field equations in stationary conducting matter, i.e. they are to satisfy equations (2·1) to (2·4) with the current density (2·5b), and are to have zero charge density within A and B, respectively. What happens now is best followed with the diagram in figure 2. Since the expression for $\mathbf{J}$ inside A is not (2·5b) but (2·5c), we have, in order to construct solutions of the field equations in A, to add an *induced field* (A'), which will have to extend outside A in order to satisfy the boundary conditions at A's surface. We shall suppose (A') to be computed as if the conductor outside A extended to infinity and were everywhere stationary. The field (A') will be completely determined by $(\mathscr{A})$; we shall write this functional dependence formally

$$(A') = (A')\,[\mathscr{A}]. \tag{2·6a}$$

Similarly, induction in B gives an induced field (B') which is completely determined by $(\mathscr{B})$. We denote this relationship by writing

$$(B') = (B')\,[\mathscr{B}]. \tag{2·6b}$$

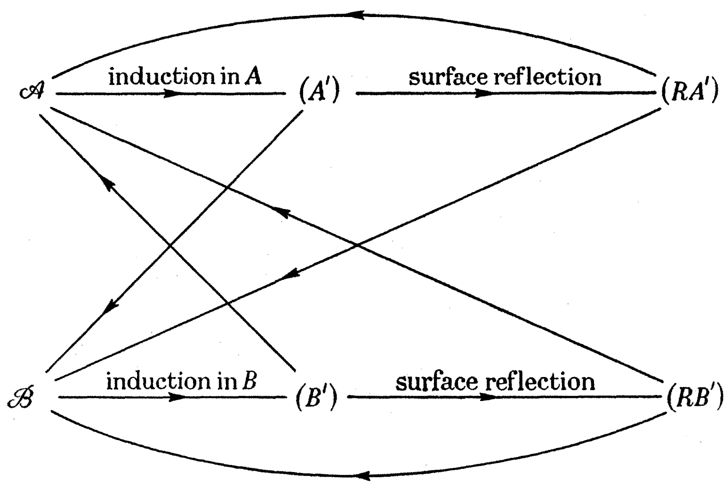

FIGURE 2

In order to satisfy the boundary conditions at $r = M$, we have now to add reflected fields (RA') and (RB') which we construct so as to satisfy (2·1) to (2·4) with $\mathbf{J}$ given by (2·5b) everywhere in $r < M$; i.e. in constructing (RA') and (RB') we ignore the rotations of A and B.

The arrangement in figure 1 acts as a dynamo if we can arrange the parameters so that

$$(\mathscr{A}) = (B') + (RA') + (RB') \tag{2·6c}$$

and

$$(\mathscr{B}) = (A') + (RA') + (RB') \tag{2·6d}$$

at A and B, respectively, i.e. if equations (2·6) have a solution. The initial fields $\mathscr{A}$ and $\mathscr{B}$ are then generated by our arrangement j acting in a self-consistent way. The analysis of these equations is the main problem of this paper. The essential point on which the analysis is based is that the axially symmetric components of ($\mathscr{A}$) and ($\mathscr{B}$) are twisted to give induced magnetic fields which are proportional to ω_A and ω_B, respectively, while the induced field due to the remainder of ($\mathscr{A}$) and ($\mathscr{B}$) is comparatively negligible when ω_A and ω_B are large. (See §4.) The analysis is further simplified by picking out from the axially symmetric component of the induced fields those parts which fall off most slowly with distance from their respective rotator. Dynamo action occurs when the amplification due to the twisting of the field by each rotator just compensates for the decrease of (A') and (B') due to the distance between the rotators. [The condition for this to occur is given by (10·2) for a special orientation of the angular velocity vectors; (in (10·2), $\eta \equiv 2a/R$).] The mathematical problem consists of deriving the condition for dynamo action, and of showing that one is justified in neglecting the components of (A') and (B') which either are not axially symmetric at their respective rotators, or if axially symmetric, fall off more rapidly with distance than the parts predominantly responsible for dynamo action. For this purpose one makes $a \ll R$. It will also be shown that the effect of the conductor surface becomes negligible if one chooses $|R_A|, |R_B| \ll M$.

The analysis of equations (2·6) will be taken up in §6 after some preliminaries which will occupy the next three sections.

3. Solutions of the field equations in stationary matter

In order to analyze equations (2·6), we shall need formulae for the fields (A'), (B'), (RA') and (RB'). It will be convenient to express these in terms of a complete set of solutions of equations (2·1) to (2·4), with $\mathbf{J}$ given by (2·5b).

From (2·2), it follows that $\mathbf{E} = \nabla\chi$, where χ is some scalar function. From (2·1) and (2·5b) we have $\operatorname{div}\mathbf{E} = 0$, so that $\nabla^2\chi = 0$. If we introduce spherical polar co-ordinates (r, θ, λ), then we can write $\chi = \partial(r\psi)/\partial r$, with $\nabla^2\psi = 0$. (2·1) and (2·3) are then satisfied by $\mathbf{H} = \sigma\nabla\psi\wedge\mathbf{r}$, as one can check by differentiating and using $\nabla^2\psi = 0$. One set of solutions of the field equations is therefore

$$\mathbf{H}_T = \sigma\nabla\psi\wedge\mathbf{r}, \tag{3·1a}$$

$$\mathbf{E}_T = \nabla\frac{\partial}{\partial r}(r\psi), \tag{3·1b}$$

$$\nabla^2\psi = 0. \tag{3·1c}$$

This mode of writing the electric field excludes the field $\mathbf{E} = \nabla(1/r)$, but such a field would have to be excluded in any case because it violates (2·1) and (2·5b); (integrate the normal component of (2·1) over a closed surface enclosing the origin).

The remaining solutions must be of the form $\mathbf{E} = 0$, $\operatorname{curl}\mathbf{H} = 0$, or

$$\mathbf{H}_S = \sigma l\nabla\psi, \tag{3·2a}$$

$$\mathbf{E}_S = 0, \tag{3·2b}$$

$$\nabla^2\psi = 0, \tag{3·2c}$$

where l has the dimensions of a length; the factor σl in (3·2a) is convenient for dimensional reasons.

The functions ψ will be written in the form

$$\xi_{nm}(r_i, \theta_i, \lambda_i) \equiv \left(\frac{r_i}{a_i}\right)^n Y_{nm}(\theta_i, \lambda_i), \quad \xi'_{nm}(r_i, \theta_i, \lambda_i) \equiv \left(\frac{r_i}{a_i}\right)^{-n-1} Y_{nm}(\theta_i, \lambda_i), \tag{3·3}$$

where m and n are integers ($m \leqslant n$, $n > 0$), and Y_{nm} are the orthonormal spherical harmonics defined by Blatt & Weisskopf (1952, p. 782). The functions (3·3) form a complete set of solutions of Laplace's equation. The definitions of a_i and $(r_i, \theta_i, \lambda_i)$, and the labels ξ and X (to be introduced in the next paragraph) will depend on the co-ordinate system in use in a way to be defined later.

The fields derived from (3·3) by the rules (3·1) and (3·2) will be denoted by the following symbols†:

$$\left.\begin{aligned}(XTnm) &\equiv \text{collective label for } (XTnm;\, \mathbf{H}) \equiv \sigma \nabla \xi_{nm} \wedge \mathbf{r}_i,\\ &\text{and } (XTnm;\, \mathbf{E}) \equiv \nabla \frac{\partial}{\partial r_i}(r_i \xi_{nm});\end{aligned}\right\} \tag{3·4a}$$

$$\left.\begin{aligned}(X'Tnm) &\equiv \text{collective label for } (X'Tnm;\, \mathbf{H}) \equiv \sigma \nabla \xi'_{nm} \wedge \mathbf{r}_i,\\ &\text{and } (X'Tnm;\, \mathbf{E}) \equiv \nabla \frac{\partial}{\partial r_i}(r_i \xi'_{nm});\end{aligned}\right\} \tag{3·4b}$$

$$\left.\begin{aligned}(XSnm) &\equiv \text{collective label for } (XSnm;\, \mathbf{H}) \equiv \sigma a_i \nabla \xi_{nm},\\ &\text{and } (XSnm;\, \mathbf{E}) \equiv 0;\end{aligned}\right\} \tag{3·4c}$$

$$\left.\begin{aligned}(X'Snm) &\equiv \text{collective label for } (X'Snm;\, \mathbf{H}) \equiv \sigma a_i \nabla \xi'_{nm},\\ &\text{and } (X'Snm;\, \mathbf{E}) \equiv 0.\end{aligned}\right\} \tag{3·4d}$$

The field functions given in (3·4) are very similar to the field functions defined by Bullard (1949, table 2). $\mathbf{H}_T$ and $\mathbf{H}_S$ are proportional to the toroidal and poloidal magnetic fields defined by Bullard. The associated electric fields are denoted by Bullard by $\mathbf{E}_S$ (poloidal) and $\mathbf{E}_T$ (toroidal), respectively. In this paper, the magnetic fields will play the predominant role, and we therefore denote the electric fields by the suffix of the magnetic field with which they are associated, i.e. in our notation the association is $(\mathbf{H}_T, \mathbf{E}_T)$ and $(\mathbf{H}_S, \mathbf{E}_S)$, instead of $(\mathbf{H}_T, \mathbf{E}_S)$ and $(\mathbf{H}_S, \mathbf{E}_T)$ as with Bullard.

The exact relations between Bullard's field functions and ours is as follows (Bullard's fields are denoted by a subscript B; unprimed functions (3·4a) and (3·4c) increase with distance from the origin, while primed functions (3·4b) and (3·4d) decrease):

$$(XTnm;\, \mathbf{H})_B = \frac{1}{n+1}\left[\frac{4\pi(n+m)!}{(2n+1)(n-m)!}\right]^{\frac{1}{2}} (XTnm;\, \mathbf{H}), \tag{3·5a}$$

$$(X'Tnm;\, \mathbf{H})_B = -\frac{1}{n}\left[\frac{4\pi(n+m)!}{(2n+1)(n-m)!}\right]^{\frac{1}{2}} (X'Tnm;\, \mathbf{H}), \tag{3·5b}$$

$$(XSnm;\, \mathbf{H})_B = \frac{1}{\sigma}\left[\frac{4\pi(n+m)!}{(2n+1)(n-m)!}\right]^{\frac{1}{2}} (XSnm;\, \mathbf{H}), \tag{3·5c}$$

$$(X'Snm;\, \mathbf{H})_B = \frac{1}{\sigma}\left[\frac{4\pi(n+m)!}{(2n+1)(n-m)!}\right]^{\frac{1}{2}} (X'Snm;\, \mathbf{H}). \tag{3·5d}$$

The relations for the electric fields follow immediately from those for the associated magnetic ones. Note that Bullard used unrationalized e.m.u. and denoted the conductivity by κ, so that $(4\pi\kappa)$ appears in his formulae instead of σ in ours.

† The notation $(\ldots;\, \mathbf{H})$ and $(\ldots;\, \mathbf{E})$ will be used to denote magnetic and electric fields of particular kinds. The dots indicate the labels specifying the character of the field.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

We shall need the solutions (3·4) for the cases where the origin is taken at O, $\mathbf{R}_A$ or $\mathbf{R}_B$ (see figure 1). In the first case (origin at O), the axes may be chosen arbitrarily; the polar angles will be denoted by θ and λ. In the second and third cases (origin at $\mathbf{R}_A$ or $\mathbf{R}_B$), the z-axes will be taken along $\boldsymbol{\omega}_A$ and $\boldsymbol{\omega}_B$, respectively, and the polar angles will be denoted by (θ_A, λ_A) and (θ_B, λ_B); the directions $\lambda_A = 0$ and $\lambda_B = 0$ may be chosen arbitrarily. Solutions of the form (3·4) for the three co-ordinate systems will be written by making the substitutions specified in table 1. As an example, the magnetic field $\mathbf{H}_T$ which has the origin at $\mathbf{R}_A$, the angular integers n and m, and which increases with r_A will be written

$$(ATnm; \mathbf{H}) \equiv \sigma \nabla \alpha_{nm} \wedge \mathbf{r}_A, \quad \text{where} \quad \alpha_{nm} \equiv \left(\frac{r_A}{a}\right)^n Y_{nm}(\theta_A, \lambda_A).$$

Table 1. Notation for the field functions

symbol in (3·4)	X	ξ	$(r_i, \theta_i, \lambda_i)$	a_i
symbol for origin at $\mathbf{R}_A$	A	α	$(r_A, \theta_A, \lambda_A)$	a
symbol for origin at $\mathbf{R}_B$	B	β	$(r_B, \theta_B, \lambda_B)$	a
symbol for origin at O	C	γ	(r, θ, λ)	M

In the course of the analysis of (2·6) it will be necessary to expand the fields $(\mathscr{A})$ and $(\mathscr{B})$ in terms of the functions (3·4). We next derive the necessary formulae. In the neighbourhood of sphere A, the completeness of the functions (3·4) permits us to write

$$(\mathscr{A}) = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \left[\begin{pmatrix} ASnm \\ \mathscr{A} \end{pmatrix} (ASnm) + \begin{pmatrix} ATnm \\ \mathscr{A} \end{pmatrix} (ATnm) \right], \tag{3·6}$$

where the quantities $\begin{pmatrix} ASnm \\ \mathscr{A} \end{pmatrix}$ and $\begin{pmatrix} ATnm \\ \mathscr{A} \end{pmatrix}$ are coefficients which we have to determine. Only unprimed functions appear in (3·6) because $(\mathscr{A})$ is regular at $r_A = 0$. The notation used for the coefficients $\begin{pmatrix} ASnm \\ \mathscr{A} \end{pmatrix}$ and $\begin{pmatrix} ATnm \\ \mathscr{A} \end{pmatrix}$ requires some comment, especially because similar two-line symbols will be introduced at a number of other places later. It is convenient to think of the field $(\mathscr{A})$ as a *cause*, which gives rise to the *effects* $(ASnm)$ and $(ATnm)$; in the coefficient $\begin{pmatrix} A\,S \text{ or } T\,nm \\ \mathscr{A} \end{pmatrix}$ describing this process, the cause appears in the lower line, and the effect in the upper. The two sides of equation (3·6) are no more than different representations of the same thing, so that to speak of cause and effect is merely a matter of convenience, because no physical mechanism is involved by which one gives rise to the other. A physical mechanism will appear later, when we come to use a similar notation in § 4 to describe the induction in a rotating sphere; the effect (in the upper line) will arise from the cause (in the lower line) through the induction process.

We can determine the coefficients $\begin{pmatrix} ASnm \\ \mathscr{A} \end{pmatrix}$ in (3·6) by noting that according to (3·4) and table 1, the magnetic component $(\mathscr{A}; \mathbf{H}).\hat{\mathbf{r}}_A$ comes entirely from $(ASnm)$. Therefore, we have from (3·3), (3·4c), (3·6) and table 1,

$$\{(\mathscr{A}; \mathbf{H}).\hat{\mathbf{r}}_A\}_{r_A=a} = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \begin{pmatrix} ASnm \\ \mathscr{A} \end{pmatrix} \sigma n Y_{nm}(\theta_A, \lambda_A),$$

so that

$$\begin{pmatrix} ASnm \\ \mathscr{A} \end{pmatrix} = \frac{1}{\sigma n} \int \mathrm{d}\Omega_A \, Y^*_{nm}(\theta_A, \lambda_A) \, \{(\mathscr{A}; \mathbf{H}).\hat{\mathbf{r}}_A\}_{r_A=a}, \tag{3·7}$$

where $\mathrm{d}\Omega_A \equiv \sin\theta_A \, \mathrm{d}\theta_A \, \mathrm{d}\lambda_A$ and where a * denotes the complex conjugate.

The coefficients $\binom{ATnm}{\mathscr{A}}$ in (3·6) can be determined by noting ((3·4), table 1) that the electric field comes entirely from $(ATnm)$. Therefore we have from (3·3), (3·4*a*), table 1 and (3·6)

$$\{(\mathscr{A};\mathbf{E}).\hat{\mathbf{r}}_A\}_{r_A=a} = \sum_{n=1}^{\infty}\sum_{m=-n}^{n}\binom{ATnm}{\mathscr{A}}\frac{n(n+1)}{a}Y_{nm}(\theta_A,\lambda_A),$$

so that
$$\binom{ATnm}{\mathscr{A}} = \frac{a}{n(n+1)}\int \mathrm{d}\Omega_A Y^*_{nm}(\theta_A,\lambda_A)\{(\mathscr{A};\mathbf{E}).\hat{\mathbf{r}}_A\}_{r_A=a}. \tag{3·8}$$

Occasionally we shall have to express not the whole of $(\mathscr{A})$, but only some part in a series of the form (3·6). In particular, this will be necessary for components of the form $(B'Snm)$ and $(B'Tnm)$. We shall next derive the relevant formulae.

We shall begin by deriving expansions for the functions β'_{nm} defined by (3·3) and table 1. Since the functions α_{nm} are a complete set of regular solutions of Laplace's equation, we may write,

$$\beta'_{nm} = \sum_{n'=1}^{\infty}\sum_{m'=-n'}^{n'}\binom{\alpha n'm'}{\beta' nm}\alpha_{n'm'}. \tag{3·9}$$

An expansion of the form (3·9) exists only when $r_A<R$, i.e. where β'_{nm} is regular.

We shall consider first the special case where the axes of the co-ordinate systems (r_A,θ_A,λ_A) and (r_B,θ_B,λ_B) are parallel; the axes $\theta_A=0$, $\theta_B=0$ will be assumed to lie along the direction $+\mathbf{R}$, and the planes $\lambda_A=0$ and $\lambda_B=0$ will be taken to be parallel. All quantities referred to these special co-ordinate systems will be distinguished by a suffix $\|$, i.e. we write in place of (3·9)

$$\beta'_{nm\|} = \sum_{n'=1}^{\infty}\sum_{m'=-n'}^{n'}\binom{\alpha n'm'}{\beta' nm}_{\|}\alpha_{n'm'\|}. \tag{3·10}$$

To determine the coefficients $(\because)_{\|}$ in (3·10), we express $\alpha_{n'm'\|}$ and $\beta'_{nm\|}$ in the form given in (3·3) and table 1, and write out

$$Y_{nm}(\theta,\lambda) = (-1)^m\left[\frac{2n+1}{4\pi}\frac{(n-m)!}{(n+m)!}\right]^{\frac{1}{2}}P_{nm}(\cos\theta)\,\mathrm{e}^{\mathrm{i}m\lambda}, \tag{3·11}$$

where P_{nm} is the associated Legendre polynomial as defined in Jahnke & Emde (1945, p. 110). From (3·11) and our special choice of co-ordinate system, it follows that $(\because)_{\|}\neq 0$ only if $m=m'$; we can therefore drop a factor $\exp(\mathrm{i}m\lambda)$ from both sides in (3·10), and let n' run from m to ∞. We next expand P_{nm} as $P_{nm}(\cos\theta)=\text{constant}\times\sin^m\theta[1+O(\sin^2\theta)]$, and combine the factors $\sin^m(\theta_A,\theta_B)$ with factors (r^m_A, r^m_B) to give each term in (3·10) a factor $\rho^m=(r_A\sin\theta_A)^m=(r_B\sin\theta_B)^m$, where ρ is the perpendicular distance from the straight line passing through $\mathbf{R}_A$ and $\mathbf{R}_B$. We then cancel the factors ρ^m, and put $\theta_A=\theta_B=\pi$. The resultant equation contains only r_A and r_B, and refers only to points on the straight line $\mathbf{R}_A\rightarrow\mathbf{R}_B$. Therefore we can put $r_B=R+r_A$, after which we expand the left-hand side in powers of r_A, and equate coefficients of equal powers. We then find

$$\left.\begin{aligned}\binom{\alpha n'm'}{\beta' nm}_{\|} &= (-1)^{n-m}(n+n')!\left(\frac{a}{R}\right)^{n+n'+1}\left[\frac{2n'+1}{2n+1}(n-m)!\,(n+m)!\,(n'-m)!\,(n'+m)!\right]^{-\frac{1}{2}}\\ &\qquad \text{if}\quad m=m',\\ &= 0\quad \text{if}\quad m\neq m'.\end{aligned}\right\} \tag{3·12}$$

By a similar argument, one finds

$$\begin{pmatrix}\beta n'm' \\ \alpha' nm\end{pmatrix}_{\|} = (-1)^{n+n'} \begin{pmatrix}\alpha n'm' \\ \beta' nm\end{pmatrix}_{\|}. \tag{3·13}$$

We now drop the restriction to *parallel* axes of the two co-ordinate systems used in (3·10). Let us suppose that the actual co-ordinate systems $(r_A, \theta_A, \lambda_A)$ and $(r_B, \theta_B, \lambda_B)$ have axes which are obtained from the 'parallel' systems by rotations with Eulerian angles $(\Theta_A, \Lambda_A, \Psi_A)$ and $(\Theta_B, \Lambda_B, \Psi_B)$, respectively; these two rotations will be denoted by T_A and T_B. Then we can write (Wigner 1931)

$$\left.\begin{aligned} Y_{nm}(\theta_A, \lambda_A) &= \sum_{m'=-n}^{n} \mathscr{D}_{n,mm'}(T_A)\, Y_{nm'}(\theta_{A\|}, \lambda_{A\|}), \\ Y_{nm}(\theta_B, \lambda_B) &= \sum_{m'=-n}^{n} \mathscr{D}_{n,mm'}(T_B)\, Y_{nm'}(\theta_{B\|}, \lambda_{B\|}), \end{aligned}\right\} \tag{3·14a}$$

so that

$$\alpha_{nm} = \sum_{m'=-n}^{n} \mathscr{D}_{n,mm'}(T_A)\, \alpha_{nm'\|}, \quad \beta_{nm} = \sum_{m'=-n}^{n} \mathscr{D}_{n,mm'}(T_B)\, \beta_{nm'\|}; \tag{3·14b}$$

These equations are still valid if we replace the α by α' and the β by β'.

Since the Y_{nm} are orthonormal, the transformation coefficients satisfy the unitarity conditions

$$\sum_{m''=-n}^{n} \mathscr{D}^{*}_{n,mm''}(T_i)\, \mathscr{D}_{n,m'm''}(T_i) = \delta_{mm'} \quad (i = A \text{ or } B). \tag{3·15}$$

By using (3·14) and (3·15), we find from (3·9) and (3·10) that

$$\begin{pmatrix}\alpha n'm' \\ \beta' nm\end{pmatrix} = \sum_{m''=-n}^{n} \mathscr{D}_{n,mm''}(T_B) \begin{pmatrix}\alpha n'm'' \\ \beta' nm''\end{pmatrix}_{\|} \mathscr{D}^{*}_{n',m'm''}(T_A). \tag{3·16}$$

A similar equation holds if one interchanges $\alpha \rightleftharpoons \beta$, $A \rightleftharpoons B$.

We are now ready to derive formulae for the expansion coefficients in the series

$$(B'Snm) = \sum_{n'=1}^{\infty} \sum_{m'=-n'}^{n'} \left\{\begin{pmatrix}ASn'm' \\ B'Snm\end{pmatrix}(ASn'm') + \begin{pmatrix}ATn'm' \\ B'Snm\end{pmatrix}(ATn'm')\right\} \tag{3·17a}$$

and

$$(B'Tnm) = \sum_{n'=1}^{\infty} \sum_{m'=-n'}^{n'} \left\{\begin{pmatrix}ASn'm' \\ B'Tnm\end{pmatrix}(ASn'm') + \begin{pmatrix}ATn'm' \\ B'Tnm\end{pmatrix}(ATn'm')\right\}. \tag{3·17b}$$

By writing $(B'Snm)$ in the form (3·4*d*), using the expansion (3·9) and the formulae (3·7) and (3·8), we get

$$\begin{pmatrix}ASn'm' \\ B'Snm\end{pmatrix} = \begin{pmatrix}\alpha n'm' \\ \beta' nm\end{pmatrix}, \quad \begin{pmatrix}ATn'm' \\ B'Snm\end{pmatrix} = 0. \tag{3·18}$$

(Replace $(\mathscr{A})$ by $(B'Snm)$ in (3·7) and (3·8).) (3·18) is valid also if one exchanges A with B and α with β.

To evaluate the coefficients in (3·17*b*), we note that we can write

$$(B'Tnm;\, \mathbf{E}) \equiv \nabla \frac{\partial}{\partial r_B}(r_B \beta'_{nm}) = \nabla \frac{\partial}{\partial r_A}(r_A \beta'_{nm}) - \nabla \mathbf{R} . \nabla \beta'_{nm}, \tag{3·19a}$$

$$(B'Tnm;\, \mathbf{H}) \equiv \sigma \nabla \beta'_{nm} \wedge \mathbf{r}_B = \sigma \nabla \beta'_{nm} \wedge \mathbf{r}_A - \sigma \nabla \beta'_{nm} \wedge \mathbf{R}. \tag{3·19b}$$

MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES
PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A

The coefficients $\begin{pmatrix} ASn'm' \\ B'Tnm \end{pmatrix}$ are obtained by inserting (3·19*b*) into (3·7) (with $(\mathscr{A}) \to (B'Tnm)$). The first term in (3·19*b*) does not contribute to the integral; in the integral arising from the second term, we express $Y^*_{n'm'}(\theta_A, \lambda_A)$ and β'_{nm} as in (3·14), and then use the expansion (3·10). Noting that

$$\mathbf{r}_A \cdot (\nabla \alpha_{nm\|} \wedge \mathbf{R}) = (\mathbf{R} \wedge \mathbf{r}_A) \,.\, \nabla \alpha_{nm\|} = R(\partial \alpha_{nm\|}/\partial \lambda_A) = Rim\alpha_{nm\|},$$

and making use of the orthonormality of the spherical harmonics to simplify the expressions, one obtains

$$\begin{pmatrix} ASn'm' \\ B'Tnm \end{pmatrix} = -\frac{Ri}{an'} \sum_{m''=-n}^{n} m'' \begin{pmatrix} \alpha n'm'' \\ \beta' nm'' \end{pmatrix}_{\|} \mathscr{D}^*_{n',m'm''}(T_A)\, \mathscr{D}_{n,mm''}(T_B). \tag{3·20}$$

The coefficients $\begin{pmatrix} BSn'm' \\ A'Tnm \end{pmatrix}$ are obtained from (3·20) by putting $A \rightleftharpoons B$, $\alpha \rightleftharpoons \beta$, and multiplying the whole expression by (-1); (the changed sign comes from (3·19)).

The coefficients (3·20) with $m = m' = 0$ will play an important role in the later development. To compute these, we need the coefficients $\mathscr{D}_{n,0m}$. These can be obtained from the addition theorem for the spherical harmonics:

$$Y_{n0}(\theta_A) = \left(\frac{4\pi}{2n+1}\right)^{\frac{1}{2}} \sum_{m=-n}^{n} Y^*_{nm}(\Theta_A, \Lambda_A)\, Y_{nm}(\theta_{A\|}, \lambda_{A\|}), \tag{3·21}$$

(see, for example, Blatt & Weisskopf 1952, p. 784). A similar formula holds if A and B are interchanged. From (3·14*a*) and (3·21), we get

$$\left.\begin{aligned} \mathscr{D}_{n,0m}(T_A) &= \left(\frac{4\pi}{2n+1}\right)^{\frac{1}{2}} Y^*_{nm}(\Theta_A, \Lambda_A), \\ \mathscr{D}_{n,0m}(T_B) &= \left(\frac{4\pi}{2n+1}\right)^{\frac{1}{2}} Y^*_{nm}(\Theta_B, \Lambda_B). \end{aligned}\right\} \tag{3·22}$$

The $\mathscr{D}_{n,m'm}$ with $m' \neq 0$ will not be required explicitly.

The coefficients $\begin{pmatrix} ATn'm' \\ B'Tnm \end{pmatrix}$ in (3·17*b*) are obtained most easily from (3·19*a*) because the electric field is entirely of the form $\mathbf{E}_T$ (by 3·2*b*). We shall write

$$\begin{pmatrix} ATn'm' \\ B'Tnm \end{pmatrix} = \begin{pmatrix} ATn'm' \\ B'Tnm \end{pmatrix}_1 + \begin{pmatrix} ATn'm' \\ B'Tnm \end{pmatrix}_2; \tag{3·23}$$

the two terms are to correspond respectively to the first and second terms in (3·19*a*). By introducing the expansion (3·9) into the first term of (3·19*a*), and using (3·8) (with $(\mathscr{A}; \mathbf{E}) \to \nabla[\partial(r_A \beta'_{nm})/\partial r_A]$), we get

$$\begin{pmatrix} ATn'm' \\ B'Tnm \end{pmatrix}_1 = \begin{pmatrix} \alpha n'm' \\ \beta' nm \end{pmatrix}. \tag{3·24}$$

The second term in (3·19*a*) may be written

$$-\mathbf{R} \,.\, \nabla \beta'_{nm} = \sum_{n'=1}^{\infty} \sum_{m'=-n'}^{n'} \begin{pmatrix} ATn'm' \\ B'Tnm \end{pmatrix}_2 \frac{\partial}{\partial r_A}(r_A \alpha_{n'm'}), \tag{3·25}$$

because the corresponding electric field is entirely of the form $\mathbf{E}_T$, and the $\alpha_{n'm'}$ form a complete set of regular solutions of Laplace's equation. The coefficients $(\vdots\,\vdots)_2$ are now determined by introducing the expansion (3·9) on the left of (3·25), multiplying both sides by $Y^*_{n'm'}(\theta_A, \lambda_A)$, and integrating over the spherical surface $r_A = a$, noting that

$$\partial(r_A \alpha_{n'm'})/\partial r_A = (n'+1)\,\alpha_{n'm'}.$$

We obtain

$$\begin{pmatrix} ATn'm' \\ B'Tnm \end{pmatrix}_2 = -\frac{1}{(n'+1)} \sum_{n''=1}^{\infty} \sum_{m''=-n''}^{n''} \begin{pmatrix} \alpha n''m'' \\ \beta' nm \end{pmatrix} \int \mathrm{d}\Omega_A\, Y^*_{n'm'}(\theta_A, \lambda_A)\, (\mathbf{R}.\nabla \alpha_{n''m''})_{r_A=a}. \tag{3·26}$$

The sum on the right of (3·26) can be simplified if we note that $\mathbf{R}.\nabla\alpha_{n''m''}$ satisfies Laplace's equation and is proportional to $r_A^{n''-1}$; it is therefore a sum of terms of the form $\alpha_{n''-1\ m'''}$, with different values of m'''. Therefore the integral in (3·26) vanishes unless $n'' = n'+1$.

By combining (3·23), (3·24) and (3·26), we get

$$\begin{pmatrix} ATn'm' \\ B'Tnm \end{pmatrix} = \begin{pmatrix} \alpha n'm' \\ \beta' nm \end{pmatrix} - \frac{1}{n'+1} \sum_{m''=-n'-1}^{n'+1} \begin{pmatrix} \alpha(n'+1)\,m'' \\ \beta' nm \end{pmatrix} \int \mathrm{d}\Omega_A\, Y^*_{n'm'}(\theta_A, \lambda_A)\, (\mathbf{R}.\nabla \alpha_{n'+1\ m''})_{r_A=a}. \tag{3·27}$$

The coefficients $\begin{pmatrix} BTn'm' \\ A'Tnm \end{pmatrix}$ can be obtained from (3·27) by exchanging $A \rightleftharpoons B$ and $\alpha \rightleftharpoons \beta$, and reversing the sign in front of the second term; (the change of sign comes from (3·19)).

4. Induction in a rotating sphere

To analyze equations (2·6*a*, *b*), we have to know the 'induced' fields (A') and (B') outside the rotators which accompany the 'applied' fields ($\mathscr{A}$) and ($\mathscr{B}$). The object of this section is to treat this problem. (Induced fields outside the rotators will be defined as being of the form (3·4*b*) and (3·4*d*) (i.e. decreasing when r increases), while applied fields are of the form (3·4*a*) and (3·4*c*) (i.e. increasing when r increases).)

The induced fields occur because inside the rotating spheres the current density is (2·5*c*) instead of (2·5*b*). They can be calculated by joining solutions of the field equations with the current density (2·5*c*) inside the rotators to solutions of the form (3·4) outside.

We shall suppose ($\mathscr{A}$) and ($\mathscr{B}$) to be expanded as in (3·6), and consider the components ($ASnm$) and ($ATnm$) separately. Let ($IASnm$) and ($IATnm$) be the induced fields due to ($ASnm$) and ($ATnm$), respectively. Then we can write

$$\left(IA\,{}^{S}_{T}nm\right) = \sum_{n'=1}^{\infty} \sum_{m'=-n'}^{n'} \left\{ \begin{pmatrix} A'Sn'm' \\ \omega \\ A\,{}^{S}_{T}nm \end{pmatrix} (A'Sn'm') + \begin{pmatrix} A'Tn'm' \\ \omega \\ A\,{}^{S}_{T}nm \end{pmatrix} (A'Tn'm') \right\}. \tag{4·1}$$

The quantities $\begin{pmatrix} \cdots \\ \omega \\ \cdots \end{pmatrix}$ are expansion coefficients; we have inserted an ω into them to remind us that we are dealing with the induction process, and not merely with a change of representation as in (3·6), (3·9) and (3·17). In the coefficients $\begin{pmatrix} \cdots \\ \omega \\ \cdots \end{pmatrix}$ a similar notation is used to that introduced in (3·6); they represent the extent to which the 'cause' (the applied field) in the bottom line gives rise to the 'effect' (the induced field) in the top line. (In the rest of this section we shall discuss rotator A; corresponding results for rotator B can be obtained by replacing A by B in the formulae.)

The induced fields ($IASnm$) have been worked out by Bullard (1949). From his equations, one gets, by taking into account the relations (3·5) ($\delta_{nn'}$ is the Kronecker symbol):

$m \neq 0$

$$\begin{pmatrix} A'Sn'm' \\ \omega \\ ASnm \end{pmatrix} = \delta_{nn'}\delta_{mm'}\left(-\frac{n}{n+1}\right)\frac{J_{n+\frac{3}{2}}(ka)}{J_{n-\frac{1}{2}}(ka)}; \tag{4·2a}$$

$$\begin{pmatrix} A'Tn'm' \\ \omega \\ ASnm \end{pmatrix} = \delta_{(n-1)n'}\delta_{mm'}\left[\frac{(2n+1)\,(n^2-m^2)}{2n-1}\right]^{\frac{1}{2}}\frac{\mathrm{i}J_{n+\frac{1}{2}}(ka)}{mJ_{n-\frac{3}{2}}(ka)} - \delta_{(n+1)n'}\delta_{mm'}\left(\frac{n}{n+1}\right)\left[\frac{(2n+1)\,((n+1)^2-m^2)}{2n+3}\right]^{\frac{1}{2}}\frac{\mathrm{i}J_{n+\frac{3}{2}}(ka)}{mJ_{n-\frac{1}{2}}(ka)}; \tag{4·2b}$$

(the term with $\delta_{(n-1)n'}$ has to be omitted when $n = 1$).

$m = 0$

$$\begin{pmatrix} A'Sn'm' \\ \omega \\ ASn0 \end{pmatrix} = 0; \tag{4·3a}$$

$$\begin{pmatrix} A'Tn'm' \\ \omega \\ ASn0 \end{pmatrix} = \delta_{(n-1)n'}\delta_{0m'}\left[\frac{2n+1}{2n-1}\right]^{\frac{1}{2}}\frac{n}{4n^2-1}\,\omega a^2\sigma - \delta_{(n+1)n'}\delta_{0m'}\left[\frac{2n+1}{2n+3}\right]^{\frac{1}{2}}\frac{n}{[(2n+2)^2-1]}\,\omega a^2\sigma; \tag{4·3b}$$

(the term with $\delta_{(n-1)n'}$ has again to be omitted when $n = 1$). k is defined by

$$k^2 \equiv -\mathrm{i}\omega\sigma m. \tag{4·4}$$

(Bullard treated the case of a rotating sphere of radius b enclosed in a spherical conducting shell of radius a. To get formulae (4·2) and (4·3), one has to put $a \to \infty$, $b \to a$ in his results.)

The case of an applied field of the form ($ATnm$) does not seem to have been treated in the literature. To find the corresponding coefficients in (4·1), it will be necessary to derive solutions of the field equations (2·1) to (2·4) inside the rotators, i.e. with the current density (2·5c).

Equations (2·2) and (2·3) are satisfied by (3·1a) and (3·1b) without any restrictions on ψ.

Considering (2·1), we have $\mathbf{v} = \omega\hat{\mathbf{z}}_A \wedge \mathbf{r}_A$ in sphere A, and $\mathbf{v}\wedge\mathbf{H} = -\omega\sigma(\partial\psi/\partial\lambda_A)\,\mathbf{r}_A$. If we now assume that $\psi \propto \exp(\mathrm{i}m\lambda_A)$ we get

$$\mathbf{v}\wedge\mathbf{H} = -\mathrm{i}m\sigma\omega\psi\mathbf{r}_A. \tag{4·5}$$

Inserting this expression into (2·5c), we find that (2·1) is satisfied provided that

$$\nabla^2\psi = \mathrm{i}m\sigma\omega\psi.$$

One set of solutions inside A is therefore

$$\mathbf{H}_T^{(\omega)} = \sigma\nabla\psi^{(\omega)}\wedge\mathbf{r}_A, \tag{4·6a}$$

$$\mathbf{E}_T^{(\omega)} = \nabla\frac{\partial}{\partial r_A}(r_A\psi^{(\omega)}), \tag{4·6b}$$

$$\nabla^2\psi^{(\omega)} - \mathrm{i}m\sigma\omega\psi^{(\omega)} = 0, \quad \psi^{(\omega)} \propto \mathrm{e}^{\mathrm{i}m\lambda_A}, \tag{4·6c}$$

where superfixes ω are introduced to emphasize that the solutions refer to the interior of the rotating region. We shall see below that the remaining interior solutions are not needed.

An interior solution (4·6) can be continued into $r_A > a$ by introducing a field (3·1) as follows: we can make **H** continuous on $r_A = a$ by putting $\psi^{(\omega)} = \psi$; and we can make $\mathbf{E} \wedge \mathbf{r}_A$ continuous by putting $[\partial(r_A \psi^{(\omega)})/\partial r_A]_{r_A=a} = [\partial(r_A \psi)/\partial r_A]_{r_A=a}$. These boundary conditions are satisfied if

$$\psi^{(\omega)} = \psi, \quad \frac{\partial \psi^{(\omega)}}{\partial r_A} = \frac{\partial \psi}{\partial r_A} \quad \text{on} \quad r_A = a. \tag{4·7}$$

Suppose that $\psi^{(\omega)} = \text{constant} \times (kr_A)^{-\frac{1}{2}} J_{n+\frac{1}{2}}(kr_A)\, Y_{nm}(\theta_A, \lambda_A)$; this satisfies (4·6$c$) and is regular at $r_A = 0$. We can satisfy (4·7) by matching this function to

$$\psi = \left[\left(\frac{r_A}{a}\right)^n + P\left(\frac{a}{r_A}\right)^{n+1}\right] Y_{nm}(\theta_A, \lambda_A),$$

where P is to be determined; after using some recurrence relations for the Bessel functions, one finds that

$$P = \frac{J_{n+\frac{3}{2}}(ka)}{J_{n-\frac{1}{2}}(ka)}. \tag{4·8a}$$

It follows from the above argument that

$$\begin{pmatrix} A'\,Tn'm' \\ \omega \\ A\,Tnm \end{pmatrix} = \delta_{nn'}\delta_{mm'} \frac{J_{n+\frac{3}{2}}(ka)}{J_{n-\frac{1}{2}}(ka)} \tag{4·8b}$$

and

$$\begin{pmatrix} A'\,Sn'm' \\ \omega \\ A\,Tnm \end{pmatrix} = 0. \tag{4·8c}$$

By letting $k \to 0$ in (4·8b), we find

$$\begin{pmatrix} A'\,Tn'm' \\ \omega \\ A\,Tn0 \end{pmatrix} = 0; \tag{4·9a}$$

a particular case of (4·8c) is

$$\begin{pmatrix} A'\,Sn'm' \\ \omega \\ A\,Tn0 \end{pmatrix} = 0. \tag{4·9b}$$

The physical significance of (4·9) is that an axially symmetric inducing field of type ($A\,Tn0$) does not give any induction effects by interaction with the motion of the rotator. This is because such a field has no magnetic component perpendicular to the velocity of motion (see (4·5)).

The coefficients specified in equations (4·2), (4·3), (4·8) and (4·9) have some important properties. If $m = 0$, i.e. if the applied field has axial symmetry, then the induced electromagnetic field is proportional to ω (see (4·3)) unless it vanishes (as in (4·9)). This means that the induced magnetic field due to an applied field ($ASn0$) of given magnitude can be made indefinitely large by raising ω. The situation for $n \neq 0$ is quite different. We shall show that the coefficients for this case (given in (4·2) and (4·8)) are bounded as $\omega \to \infty$. This means that no matter how large ω may be, the induced magnetic field due to an applied field with $m \neq 0$ can never exceed a limit set by the magnitude of the applied field and the radius of the rotating sphere. Particular cases of this behaviour have been discussed by Bullard (1949) and Herzenberg & Lowes (1957).

We now show that the coefficients in (4·2) and (4·8) are bounded. To prove this we have to demonstrate the boundedness of $[J_{n+\frac{3}{2}}(ka)/J_{n-\frac{1}{2}}(ka)]$ as a function of ω.

It does not matter which root of k^2 we choose, because the ratios $J_{n+\frac{3}{2}}/J_{n-\frac{1}{2}}$ depend only on k^2. Neither does it matter whether the coefficient of $(-\mathrm{i})$ in k^2 (see (4·4)) is positive or negative, for a change of sign here (which could come from a change of sign in ωm) would send the two values of k into their complex conjugates and leave the modulus of the ratio $J_{n+\frac{3}{2}}/J_{n-\frac{1}{2}}$ unchanged. We shall therefore suppose that $ka = |(\omega a^2 \sigma m)^{\frac{1}{2}}| \exp(-\frac{1}{4}\mathrm{i}\pi)$; we shall denote this quantity by $u\exp(-\frac{1}{4}\mathrm{i}\pi)$, where u is real, positive, and proportional to $\omega^{\frac{1}{2}}$.

It will be convenient to consider not the ratios $J_{n+\frac{3}{2}}/J_{n-\frac{1}{2}}$, but instead the ratios $J_{n+\frac{1}{2}}/J_{n-\frac{1}{2}}$. For $n = 1$, we have

$$\frac{J_{\frac{3}{2}}(z)}{J_{\frac{1}{2}}(z)} = -\cot z + \frac{1}{z}. \tag{4·10}$$

By straightforward computation, one finds that for $z = u\exp(-\frac{1}{4}\mathrm{i}\pi)$, the modulus of this function increases monotonically from 0 to 1 as u increases from zero to infinity, and is therefore bounded. We shall use this result as a majorant for the ratios with $n > 1$.

By the Mittag–Leffler theorem (for details see Jeffreys & Jeffreys 1950, p. 383), we can write

$$\frac{J_{n+\frac{1}{2}}(z)}{J_{n-\frac{1}{2}}(z)} = \sum_{j=-\infty}^{\infty}\left[\frac{J_{n+\frac{1}{2}}(a_n^{(j)})/J'_{n-\frac{1}{2}}(a_n^{(j)})}{z-a_n^{(j)}}\right], \tag{4·11}$$

where $a_n^{(j)}$ is the jth zero of $J_{n-\frac{1}{2}}(z)$; the zero at $z = 0$ is to be excluded. By using the recurrence relation

$$J'_{n-\frac{1}{2}}(z) = \frac{n-\frac{1}{2}}{z} J_{n-\frac{1}{2}}(z) - J_{n+\frac{1}{2}}(z),$$

we can replace the numerators in the sum by (-1). Putting $z = u\exp(-\frac{1}{4}\mathrm{i}\pi)$, and noting that the $a_n^{(j)}$ are real (Watson 1944, §15·25) and occur in pairs of equal magnitudes but opposite signs, we get

$$\frac{\mathrm{e}^{\frac{1}{4}\mathrm{i}\pi} J_{n+\frac{1}{2}}(u\,\mathrm{e}^{-\frac{1}{4}\mathrm{i}\pi})}{J_{n-\frac{1}{2}}(u\,\mathrm{e}^{-\frac{1}{4}\mathrm{i}\pi})} = 2u\sum_{j=1}^{\infty}\left(\frac{a_n^{(j)2}}{u^4+a_n^{(j)4}}\right) - \mathrm{i}\,.\,2u^3\sum_{j=1}^{\infty}\left(\frac{1}{u^4+a_n^{(j)4}}\right). \tag{4·12}$$

We now use (4·10) as a majorant for (4·12). The only fact we need for the comparison is that $a_n^{(j)}$ increases monotonically with n for a fixed value of j (Watson 1944, §15·22).

In the imaginary part of the expression on the right of (4·12), each term in the sum decreases monotonically as the corresponding $a_n^{(j)}$ increases; it follows that

$$0 > \mathscr{I}\left[\frac{\mathrm{e}^{\frac{1}{4}\mathrm{i}\pi} J_{n+\frac{1}{2}}(u\,\mathrm{e}^{-\frac{1}{4}\mathrm{i}\pi})}{J_{n-\frac{1}{2}}(u\,\mathrm{e}^{-\frac{1}{4}\mathrm{i}\pi})}\right] > \mathscr{I}\left[\frac{\mathrm{e}^{\frac{1}{4}\mathrm{i}\pi} J_{\frac{3}{2}}(u\,\mathrm{e}^{-\frac{1}{4}\mathrm{i}\pi})}{J_{\frac{1}{2}}(u\,\mathrm{e}^{-\frac{1}{4}\mathrm{i}\pi})}\right]. \tag{4·13a}$$

We next consider the real part of the right-hand side of (4·12). We compare each term in (4·12) which has $|a_n^{(j)}| < u$ $(> u)$ with that term in the series for $n = 1$ which has its $|a_1^{(j)}|$ next above (below) $|a_n^{(j)}|$; in each pair of terms, that with $n > 1$ is less than that with $n = 1$, because, for a particular value of u, the quantity $(a_n^{(j)2}/(u^4+a_n^{(j)4}))$ increases with $a_n^{(j)}$ when $a_n^{(j)} < u$, and decreases when $a_n^{(j)} > u$. With one possible exception, each term in the series with $n = 1$ is used only once in the comparisons because the spacing of the zeros $a_n^{(j)}$ is greater than that of the $a_1^{(j)}$ at all points of the real z-axis (this follows from the theorem given by Watson 1944, §15·83). The possible exception is a term with $a_1^{(j)} \approx u$ which may have been used twice. If there is such a term, remove the term in (4·12) with $a_n^{(j)}$ next to and above u

from the comparisons and estimate it separately. It is less than the maximum of $2ua^2/(a^4+u^4)$ as a function of a, i.e. less than $(1/u)$. Such a term can occur only if $u>|a_1^{(1)}|=\pi$ and is therefore less than π^{-1}. Finally, one gets

$$0<\mathscr{R}\left[\frac{e^{\frac{1}{4}i\pi}J_{n+\frac{1}{2}}(u\,e^{-\frac{1}{4}i\pi})}{J_{n-\frac{1}{2}}(u\,e^{-\frac{1}{4}i\pi})}\right]<\mathscr{R}\left[\frac{e^{\frac{1}{4}i\pi}J_{\frac{3}{2}}(u\,e^{-\frac{1}{4}i\pi})}{J_{\frac{1}{2}}(u\,e^{-\frac{1}{4}i\pi})}\right]+\frac{1}{\pi}. \tag{4·13b}$$

From (4·13a,b) it follows that

$$\left|\frac{J_{n+\frac{1}{2}}(u\,e^{-\frac{1}{4}i\pi})}{J_{n-\frac{1}{2}}(u\,e^{-\frac{1}{4}i\pi})}\right|^2=\left[\mathscr{R}\left(\frac{e^{\frac{1}{4}i\pi}J_{n+\frac{1}{2}}}{J_{n-\frac{1}{2}}}\right)\right]^2+\left[\mathscr{I}\left(\frac{e^{\frac{1}{4}i\pi}J_{n+\frac{1}{2}}}{J_{n-\frac{1}{2}}}\right)\right]^2<\left[\mathscr{R}\left(\frac{e^{\frac{1}{4}i\pi}J_{\frac{3}{2}}}{J_{\frac{1}{2}}}\right)\right]^2+\left[\mathscr{I}\left(\frac{e^{\frac{1}{4}i\pi}J_{\frac{3}{2}}}{J_{\frac{1}{2}}}\right)\right]^2$$
$$+\frac{2}{\pi}\mathscr{R}\left(\frac{e^{\frac{1}{4}i\pi}J_{\frac{3}{2}}}{J_{\frac{1}{2}}}\right)+\frac{1}{\pi^2}<1+\frac{2}{\pi}+\frac{1}{\pi^2}. \tag{4·14}$$

Finally, we have from (4·14)

$$\left|\frac{J_{n+\frac{3}{2}}(u\,e^{-\frac{1}{4}i\pi})}{J_{n-\frac{1}{2}}(u\,e^{-\frac{1}{4}i\pi})}\right|=\left|\frac{J_{n+\frac{3}{2}}}{J_{n+\frac{1}{2}}}\right|\left|\frac{J_{n+\frac{1}{2}}}{J_{n-\frac{1}{2}}}\right|<\left(1+\frac{1}{\pi}\right)^2. \tag{4·15}$$

This completes the proof of the boundedness of the coefficients in (4·2) and (4·8).

5. The conductor surface

In §§ 3 and 4 we prepared the way for an analysis of equations (2·6) by considering series expansions of the electromagnetic fields, and the induction effects at the rotators. To complete the preparation, we next discuss the reflected fields (RA') and (RB') introduced in § 2. These fields were defined as satisfying the field equations (2·1) to (2·4) with (2·5b) in $r<M$; the fields $(A')+(RA')$ and $(B')+(RB')$ are to satisfy the boundary conditions on $r=M$.

These boundary conditions are

$$\mathbf{H}\text{ continuous},\quad \hat{\mathbf{r}}\wedge\mathbf{E}\text{ continuous},\quad \hat{\mathbf{r}}.\mathbf{E}=0\text{ on the inside of the spherical surface }r=M. \tag{5·1}$$

The first two boundary conditions are derived in the usual way from (2·1) to (2·5). The third follows from (2·1) and (2·5a,b) if one integrates the normal component of (2·1) over the surface of a disk-shaped region enclosing a section of the conductor surface.

To construct (RA'), it is convenient to expand (A') into a series of functions $(A'Tnm)$ and $(A'Snm)$, as defined in (3·4) and table 1.

The functions $(A'Snm)$ are not associated with current (according to (3·2)), and are therefore unaffected by the boundary of the conductor.

To find the fields which have to be added to $(A'Tnm)$ to satisfy the boundary conditions at $r=M$, we write

$$(A'Tnm)=\sum_{n'=1}^{\infty}\sum_{m'=-n'}^{n'}\left[\begin{pmatrix}C'Tn'm'\\A'Tnm\end{pmatrix}(C'Tn'm')+\begin{pmatrix}C'Sn'm'\\A'Tnm\end{pmatrix}(C'Sn'm')\right]. \tag{5·2}$$

This expansion is possible in $r>R_A$, where the functions $(C'Tn'm')$ and $(C'Sn'm')$ form a complete set of solutions of the field equations (2·1) to (2·4) with (2·5b).

The functions $(C'Sn'm')$ are not affected by the conductor boundary. But the functions $(C'Tn'm')$ do not satisfy the field equations in $r>M$ and therefore lead to a reflected field. The component $\hat{\mathbf{r}}.(C'Tn'm';\mathbf{E})$ is $\partial^2(r\gamma'_{n'm'})/\partial r^2$; this can be compensated on $r=M$ by a field $-(CTn'm')$. The magnetic field of the sum of these two is then

$$\sigma\nabla(\gamma'_{n'm'}-\gamma_{nm})\wedge\mathbf{r},$$

which vanishes on $r = M$. The magnetic boundary condition for the field built up starting from $(C'Tn'm')$ is therefore satisfied if we put $\mathbf{H} = 0$ in $r > M$. The electric boundary condition $(\hat{\mathbf{r}} \wedge \mathbf{E})$ continuous is satisfied if we introduce in $r > M$ an irrotational electric field which vanishes at infinity and joins on to the irrotational electric field $(C'Tn'm'; \mathbf{E}) - (CTn'm'; \mathbf{E})$ at $r = M$. The combination of an irrotational electric field and a vanishing magnetic field satisfies the field equations in $r > M$.

The reflected field corresponding to $(A'Tnm)$ is now

$$(RA'Tnm) = -\sum_{n'=1}^{\infty} \sum_{m'=-n'}^{n'} \begin{pmatrix} C'Tn'm' \\ A'Tnm \end{pmatrix} (CTn'm'). \tag{5·3}$$

In order to compute the induction effects of the reflected fields, we shall have to expand the field (5·3) into a series of the form

$$(RA'Tnm) = \sum_{n'=1}^{\infty} \sum_{m'=-n'}^{n'} \left[\begin{pmatrix} BTn'm' \\ RA'Tnm \end{pmatrix} (BTn'm') + \begin{pmatrix} BSn'm' \\ RA'Tnm \end{pmatrix} (BSn'm') \right], \tag{5·4a}$$

or the same thing with B replaced by A, or A' by B', or both. To evaluate the expansion coefficients, it is convenient to introduce the expansion

$$(CTnm) = \sum_{n'=1}^{\infty} \sum_{m'=-n'}^{n'} \left[\begin{pmatrix} BTn'm' \\ CTnm \end{pmatrix} (BTn'm') + \begin{pmatrix} BSn'm' \\ CTnm \end{pmatrix} (BSn'm') \right]. \tag{5·4b}$$

From (5·3) and (5·4b) we find that the expansion coefficients in (5·4a) are

$$\begin{pmatrix} B \begin{smallmatrix} T \\ S \end{smallmatrix} n'm' \\ RA'Tnm \end{pmatrix} = -\sum_{n''=1}^{\infty} \sum_{m''=-n''}^{n''} \begin{pmatrix} C'Tn''m'' \\ A'Tnm \end{pmatrix} \begin{pmatrix} B \begin{smallmatrix} T \\ S \end{smallmatrix} n'm' \\ CTn''m'' \end{pmatrix}. \tag{5·5}$$

The computation of the coefficients in (5·5) proceeds analogously to that of the coefficients in (3·17). We start by introducing the expansions

$$\alpha'_{nm} = \sum_{n'=1}^{\infty} \sum_{m'=-n'}^{n'} \begin{pmatrix} \gamma' n'm' \\ \alpha' nm \end{pmatrix} \gamma'_{n'm'} \tag{5·6a}$$

and

$$\gamma_{nm} = \sum_{n'=1}^{\infty} \sum_{m'=-n'}^{n'} \begin{pmatrix} \beta n'm' \\ \gamma nm \end{pmatrix} \beta_{n'm'}. \tag{5·6b}$$

In the particular system of co-ordinates in which the axes $\theta_A = 0$ and $\theta = 0$ lie along $+\mathbf{R}_A$, and in which $\lambda_A = \lambda$, we denote the coefficients in (5·6a) by a suffix $\|$; the corresponding co-ordinates will be denoted by $\theta^{(A)}_{A\|}$, $\lambda^{(A)}_{A\|}$, $\theta^{(A)}_{\|}$, $\lambda^{(A)}_{\|}$. Similarly, we denote the coefficients in (5·6b) by a suffix $\|$ in the co-ordinate system in which the axes $\theta = 0$ and $\theta_B = 0$ lie along $+\mathbf{R}_B$, and in which $\lambda_B = \lambda$; the corresponding co-ordinates will be denoted by $\theta^{(B)}_{B\|}$, $\lambda^{(B)}_{B\|}$, $\theta^{(B)}_{\|}$, $\lambda^{(B)}_{\|}$.

By a calculation similar to that which led to (3·12), one finds

$$\begin{pmatrix} \gamma' n'm' \\ \alpha' nm \end{pmatrix}_{\|} = \left(\frac{a}{M}\right)^{n+1} \left(\frac{R_A}{M}\right)^{n'-n} \frac{1}{(n'-n)!} \left[\frac{(2n+1)\,(n'+m)!\,(n'-m)!}{(2n'+1)\,(n+m)!\,(n-m)!}\right]^{\frac{1}{2}}, \tag{5·7a}$$

$$\begin{pmatrix} \beta n'm' \\ \gamma nm \end{pmatrix}_{\|} = \left(\frac{a}{M}\right)^{n'} \left(\frac{R_B}{M}\right)^{n-n'} \frac{1}{(n-n')!} \left[\frac{(2n+1)\,(n+m)!\,(n-m)!}{(2n'+1)\,(n'+m)!\,(n'-m)!}\right]^{\frac{1}{2}} \tag{5·7b}$$

if $m = m'$, and zero otherwise. The same formulae hold if one exchanges $A \rightleftharpoons B$ and $\alpha \rightleftharpoons \beta$, provided that the suffix $\|$ is suitably redefined.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

We now drop the restriction to the special co-ordinate systems used in (5·7). Let us suppose that the actual co-ordinate systems $(r_A, \theta_A, \lambda_A)$ and (r, θ, λ) have axes which are obtained from those of the systems $(r_A, \theta^{(A)}_{A\|}, \lambda^{(A)}_{A\|})$ and $(r, \theta^{(A)}_{\|}, \lambda^{(A)}_{\|})$ by rotations with Eulerian angles $(\Theta^{(A)}_A, \Lambda^{(A)}_A, \Psi^{(A)}_A)$ and $(\Theta^{(A)}, \Lambda^{(A)}, \Psi^{(A)})$, respectively; these rotations will be denoted by $T^{(A)}_A$ and $T^{(A)}$. Similarly, let us suppose that the actual co-ordinate systems $(r_B, \theta_B, \lambda_B)$ and (r, θ, λ) have axes which are obtained from those of the co-ordinate systems $(r_B, \theta^{(B)}_{B\|}, \lambda^{(B)}_{B\|})$ and $(r, \theta^{(B)}_{\|}, \lambda^{(B)}_{\|})$ by rotations with Eulerian angles $(\Theta^{(B)}_B, \Lambda^{(B)}_B, \Psi^{(B)}_B)$ and $(\Theta^{(B)}, \Lambda^{(B)}, \Psi^{(B)})$, respectively; these rotations will be denoted by $T^{(B)}_B$ and $T^{(B)}$, respectively.

We define transformation matrices analogous to those in (3·13) by the equations

$$Y_{nm}(\theta_A, \lambda_A) = \sum_{m'=-n}^{n} \mathscr{D}_{n,mm'}(T^{(A)}_A)\, Y_{nm'}(\theta^{(A)}_{A\|}, \lambda^{(A)}_{A\|}), \tag{5·8a}$$

$$Y_{nm}(\theta, \lambda) = \sum_{m'=-n}^{n} \mathscr{D}_{n,mm'}(T^{(A)})\, Y_{nm'}(\theta^{(A)}_{\|}, \lambda^{(A)}_{\|}) \tag{5·8b}$$

and

$$Y_{nm}(\theta_B, \lambda_B) = \sum_{m'=-n}^{n} \mathscr{D}_{n,mm'}(T^{(B)}_B)\, Y_{nm'}(\theta^{(B)}_{B\|}, \lambda^{(B)}_{B\|}), \tag{5·9a}$$

$$Y_{nm}(\theta, \lambda) = \sum_{m'=-n}^{n} \mathscr{D}_{n,mm'}(T^{(B)})\, Y_{nm'}(\theta^{(B)}_{\|}, \lambda^{(B)}_{\|}). \tag{5·9b}$$

These $\mathscr{D}$'s satisfy unitarity conditions like (3·15).

With the aid of the $\mathscr{D}$'s, we can write, analogously to (3·16),

$$\begin{pmatrix}\gamma' n' m' \\ \alpha' n m\end{pmatrix} = \sum_{m''=-n}^{n} \mathscr{D}_{n,mm''}(T^{(A)}_A) \begin{pmatrix}\gamma' n' m'' \\ \alpha' n m''\end{pmatrix}_{\|} \mathscr{D}^{*}_{n',m'm''}(T^{(A)}), \tag{5·10a}$$

$$\begin{pmatrix}\beta n' m' \\ \gamma n m\end{pmatrix} = \sum_{m''=-n}^{n} \mathscr{D}_{n,mm''}(T^{(B)}) \begin{pmatrix}\beta n' m'' \\ \gamma n m''\end{pmatrix}_{\|} \mathscr{D}^{*}_{n',m'm''}(T^{(B)}_B). \tag{5·10b}$$

Expressions for the factors of the coefficients in (5·5) will now be given. Analogously to (3·27), we find

$$\begin{pmatrix}C' T n' m' \\ A' T n m\end{pmatrix} = \begin{pmatrix}\gamma' n' m' \\ \alpha' n m\end{pmatrix} + \frac{1}{n'} \sum_{m''=-n'+1}^{n'-1} \begin{pmatrix}\gamma' (n'-1) m'' \\ \alpha' n m\end{pmatrix} \int \mathrm{d}\Omega Y^{*}_{n'm'}(\theta, \lambda)\, [\mathbf{R}_A . \nabla \gamma'_{(n'-1)m''}]_{r=M} \tag{5·11}$$

and

$$\begin{pmatrix}B T n' m' \\ C T n m\end{pmatrix} = \begin{pmatrix}\beta n' m' \\ \gamma n m\end{pmatrix} + \frac{1}{n'+1} \sum_{m''=-n'-1}^{n'+1} \begin{pmatrix}\beta (n'+1) m'' \\ \gamma n m\end{pmatrix} \int \mathrm{d}\Omega_B Y^{*}_{n'm'}(\theta_B, \lambda_B)\, [\mathbf{R}_B . \nabla \beta_{(n'+1)m''}]_{r_B=a}. \tag{5·12}$$

Analogously to (3·20), we find

$$\begin{pmatrix}B S n' m' \\ C T n m\end{pmatrix} = \frac{\mathrm{i} R_B}{n' a} \sum_{m''=-n'}^{n'} m'' \begin{pmatrix}\beta n' m'' \\ \gamma n m''\end{pmatrix}_{\|} \mathscr{D}^{*}_{n',m'm''}(T^{(B)}_B)\, \mathscr{D}_{n,mm''}(T^{(B)}). \tag{5·13}$$

The formulae in (5·11), (5·12) and (5·13) are valid also if one interchanges $A \rightleftharpoons B$ and $\alpha \rightleftharpoons \beta$.

An exact evaluation of the series (5·5) will not be necessary, and for most of the later work we shall be able to use upper bounds which will be derived in § 7. However, in the special case of the coefficients $\begin{pmatrix}A \text{ or } B\, S n 0 \\ R A' T n 0\end{pmatrix}$ something more precise will be needed. To compute these coefficients, we note than when $M \gg (|\mathbf{R}_A|, |\mathbf{R}_B|)$, the term in the series (5·5) with the lowest value of n'' is predominant, because, as one can show from (5·11), (5·13), (5·10) and (5·7), the terms in (5·5) contain n'' only in the form of finite powers and a factor $(|\mathbf{R}_A|\,|\mathbf{R}_B|/M^2)^{n''}$. (This statement can be made rigorous by the use of the sort of argument given in § 7, especially of (7·11).) The first non-vanishing term in the coefficients $\begin{pmatrix} A \text{ or } B\, Sn0 \\ RA'\,Tn0 \end{pmatrix}$ has $n'' = n$ because of (5·7), and contains no contribution from the second term in (5·11), again because of (5·7). We therefore have

$$\begin{pmatrix} BSn0 \\ RA'\,Tn0 \end{pmatrix} = -\frac{\mathrm{i}\,|\mathbf{R}_B|}{na}\left(\frac{a}{M}\right)^{2n+1}$$
$$\times \sum_{m=-n}^{n}\sum_{m'=-n}^{n}\sum_{m''=-n}^{n} m''\mathscr{D}_{n,0m}(T_A^{(A)})\,\mathscr{D}^*_{n,m'm}(T^{(A)})\,\mathscr{D}^*_{n,0m''}(T_B^{(B)})\,\mathscr{D}_{n,m'm''}(T^{(B)})$$
$$+\text{smaller terms containing additional factors } (|\mathbf{R}_A|\,|\mathbf{R}_B|/M^2). \qquad (5·14)$$

According to (3·22) and (3·11), the factor $\mathrm{i}m''$ can be replaced by $\partial/\partial\Lambda_B^{(B)}$ and taken outside the summation signs. Moreover, by using the unitarity of the $\mathscr{D}$'s and the fact that they form a representation of the rotation group, we have

$$\mathscr{D}^*_{n,m'm}(T^{(A)}) \equiv [\mathscr{D}_n^{+}(T^{(A)})]_{mm'} = [\mathscr{D}_n^{-1}(T^{(A)})]_{mm'} = \mathscr{D}_{n,mm'}(T^{(A)-1}), \qquad (5·15a)$$

and similarly

$$\mathscr{D}^*_{n,0m''}(T_B^{(B)}) = \mathscr{D}_{n,m''0}(T_B^{(B)-1}). \qquad (5·15b)$$

(The superscripts + and −1 denote 'Hermitian conjugate' and 'inverse' respectively.) Substituting (5·15) into (5·14), we get

$$\begin{pmatrix} BSn0 \\ RA'\,Tn0 \end{pmatrix} = -\frac{|\mathbf{R}_B|}{na}\left(\frac{a}{M}\right)^{2n+1}\frac{\partial}{\partial\Lambda_B^{(B)}}\mathscr{D}_{n,00}(T_A^{(A)}\times T^{(A)-1}\times T^{(B)}\times T_B^{(B)-1}) + \text{s.t.} \qquad (5·16)$$

Now the rotation $(T_A^{(A)}\times T^{(A)-1}\times T^{(B)}\times T_B^{(B)-1})$ rotates $\hat{\boldsymbol{\omega}}_B$ into $\hat{\boldsymbol{\omega}}_A$, so that ((3·11), (3·22))

$$\mathscr{D}_{n,00}(T_A^{(A)}\times T^{(A)-1}\times T^{(B)}\times T_B^{(B)-1}) = P_n(\hat{\boldsymbol{\omega}}_A\,.\,\hat{\boldsymbol{\omega}}_B). \qquad (5·17)$$

Moreover, it follows from the definition of $\Lambda_B^{(B)}$ that

$$\frac{\partial\hat{\boldsymbol{\omega}}_B}{\partial\Lambda_B^{(B)}} = \frac{\mathbf{R}_B\wedge\hat{\boldsymbol{\omega}}_B}{|\mathbf{R}_B|}. \qquad (5·18)$$

With the aid of (5·17) and (5·18), (5·16) becomes

$$\begin{pmatrix} BSn0 \\ RA'\,Tn0 \end{pmatrix} = -\frac{1}{n}\left(\frac{a}{M}\right)^{2n}\frac{\mathrm{d}P_n(\hat{\boldsymbol{\omega}}_A\,.\,\hat{\boldsymbol{\omega}}_B)}{\mathrm{d}(\hat{\boldsymbol{\omega}}_A\,.\,\hat{\boldsymbol{\omega}}_B)}\,\frac{\mathbf{R}_B\,.\,(\hat{\boldsymbol{\omega}}_B\wedge\hat{\boldsymbol{\omega}}_A)}{M} + \text{s.t.} \qquad (5·19)$$

The coefficient $\begin{pmatrix} ASn0 \\ RA'\,Tn0 \end{pmatrix}$ is obtained by replacing B by A in (5·19). It follows that the leading term in $\begin{pmatrix} ASn0 \\ RA'\,Tn0 \end{pmatrix}$ vanishes.

So far this section has been concerned with the calculation of the fields reflected at the conductor boundary. Another aspect of the boundary with which we shall have to deal is the calculation of the part of the magnetic component of (A') which gets out into $r > M$. This can be done by noting that the component of this field in the direction $\hat{\mathbf{r}}$ is just $\hat{\mathbf{r}}\,.\,(A', \mathbf{H})$, because this component comes entirely from the terms $(C'S\ldots)$ in (5·2) which are not affected by the boundary at $r = M$. Since the magnetic field in $r > M$ is irrotational and vanishes as $r \to \infty$, it can be written

$$\mathbf{H} = -\nabla \Psi,$$

where
$$\Psi(r, \theta, \lambda) = \int_{\infty}^{r} \frac{\partial \Psi}{\partial r'}(r', \theta, \lambda)\, dr' = -\int_{\infty}^{r} \hat{\mathbf{r}} \,.\, (A', \mathbf{H})\, dr'. \tag{5·20}$$

It is important to note that (5·20) can be used without (A')'s being first analyzed in the form (5·2).

6. Analysis of equations (2·6)

After the preliminaries of the last three sections, we can now start to analyze equations (2·6) to see whether they can have a non-vanishing solution. As a first step we expand (A') and (B') in the form

$$(A') = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} [a_{Snm}(A'Snm) + a_{Tnm}(A'Tnm)], \tag{6·1a}$$

$$(B') = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} [b_{Snm}(B'Snm) + b_{Tnm}(B'Tnm)]. \tag{6·1b}$$

These expansions are valid in the regions $r_A > a$ and $r_B > b$, respectively.

We next express the functional dependence in (2·6a, b) in terms of the a and b. If the field $(\mathscr{A})$ is expressed in the form (3·6), then each component $(ASnm)$ and $(ATnm)$ gives rise to an induced field as described by (4·1); a_{Snm} and a_{Tnm}, respectively, are the coefficients of $(A'Snm)$ and $(A'Tnm)$ in the complete induced field; i.e. we have in place of (2·6a, b)

$$a_{\substack{S\\T}nm} = \sum_{n'=1}^{\infty} \sum_{m'=-n'}^{n'} \left[\begin{pmatrix} ASn'm' \\ \mathscr{A} \end{pmatrix} \begin{pmatrix} A' \, {}^{S}_{T} nm \\ \omega \\ ASn'm' \end{pmatrix} + \begin{pmatrix} ATn'm' \\ \mathscr{A} \end{pmatrix} \begin{pmatrix} A' \, {}^{S}_{T} nm \\ \omega \\ ATn'm' \end{pmatrix} \right], \tag{6·2a}$$

$$b_{\substack{S\\T}nm} = \sum_{n'=1}^{\infty} \sum_{m'=-n'}^{n'} \left[\begin{pmatrix} BSn'm' \\ \mathscr{B} \end{pmatrix} \begin{pmatrix} B' \, {}^{S}_{T} nm \\ \omega \\ BSn'm' \end{pmatrix} + \begin{pmatrix} BTn'm' \\ \mathscr{B} \end{pmatrix} \begin{pmatrix} B' \, {}^{S}_{T} nm \\ \omega \\ BTn'm' \end{pmatrix} \right]. \tag{6·2b}$$

The significance of the coefficients $\begin{pmatrix} A \text{ or } B\ S \text{ or } T\ n'm' \\ \mathscr{A} \text{ or } \mathscr{B} \end{pmatrix}$ was explained when they were introduced in (3·6). The coefficients $\begin{pmatrix} \cdots \\ \omega \\ \cdots \end{pmatrix}$ describe the induction in the two rotators; they signify the extent to which the field in the top line (i.e. the effect) is induced when there is applied the field written in the bottom line (i.e. the cause).

These equations may be rewritten in the form

$$a_{\substack{S\\T}nm} = \sum_{n'=1}^{\infty} \sum_{m'=-n'}^{n'} \left\{ b_{Sn'm'} \begin{pmatrix} A' \, {}^{S}_{T} nm \\ \omega \\ B'Sn'm' \end{pmatrix} + b_{Tn'm'} \left[\begin{pmatrix} A' \, {}^{S}_{T} nm \\ \omega \\ B'Tn'm' \end{pmatrix} + \begin{pmatrix} A' \, {}^{S}_{T} nm \\ \omega \\ RB'Tn'm' \end{pmatrix} \right] + a_{Tn'm'} \begin{pmatrix} A' \, {}^{S}_{T} nm \\ \omega \\ RA'Tn'm' \end{pmatrix} \right\}, \tag{6·3}$$

and a set of similar equations for the b. The coefficients $\begin{pmatrix} \cdot\cdot\cdot \\ \omega^* \\ \cdot\cdot\cdot \end{pmatrix}$ in (6·3) have a meaning resembling that of the coefficients $\begin{pmatrix} \cdot\cdot\cdot \\ \omega \\ \cdot\cdot\cdot \end{pmatrix}$ introduced in (4·1). They describe how the 'cause' (in the bottom line) applied to rotator A induces the 'effect' (in the top line). The notation $\begin{smallmatrix} S \\ T \end{smallmatrix}$ means that either S or T is to be read throughout the equations. An ω has been written in each of the coefficients to remind us that induction in rotator A intervenes as one step in the generation of the a. There are no terms

$$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ & \omega & & \\ RA' \text{ or } & RB' & S\,n' & m' \end{pmatrix}$$

because S-type fields are not reflected at the conductor boundary (see § 5).

The coefficients in (6·3) can be constructed from (2·6c,d), (6·1), (3·17), the method of constructing (RA') and (RB') described in § 5, and the induction coefficients given in (4·2), (4·3), (4·8) and (4·9). The result is given in table 2.

The condition that our model should be able to act as a dynamo is that equations (6·3) should have a solution for some real value of ω. The investigation of this question will occupy most of the rest of this paper.

As a first step in the discussion, we need some orientation about which of the a's and b's are important. One way of getting such an orientation is to take into account only one pair (a_{Snm}, b_{Snm}) or (a_{Tnm}, b_{Tnm}), and to put $M = \infty$. The discussion need not be given in detail. One finds that if $m \neq 0$, then the only solution of (6·3) for large R is $a = b = 0$, because the saturation effect expressed in (4·15) makes it impossible for the amplification due to the induction process to compensate for the fall-off of the induced field with R. In contrast, one finds that for $m = 0$ the amplification at induction *can* compensate for the fall-off with R (for the T fields (but not for the S)); it turns out that a necessary condition for this compensation is $\omega \propto R^3$ for $n = 1$ or 2, while ω is proportional to a higher power of R for $n > 2$. Values of ω which satisfy this condition for $n = 1$ or 2 would in general be too small for $n > 2$.

We are therefore led to the suggestion that when M and R are large, the coefficients a_{T10}, a_{T20}, b_{T10} and b_{T20} are of paramount importance; the following treatment will be guided by this suggestion. Henceforth we shall call these four coefficients strong, and refer to the other a's and b's as the weak coefficients. The physical picture which corresponds to the predominance of the strong coefficients is that each rotator lies essentially in the other's axially symmetric toroidal magnetic field (see § 3). The appearance of the fields which finally result is described in § 11 (f).

Corresponding to our separation of the a's and b's, we shall also separate equations (6·3) into two groups: those equations (6·3) which have one of the four strong coefficients on the left will be called strong; the remainder will be called weak.

Our method of treatment of (6·3) will be as follows. We shall first solve the weak equations for the weak a's and b's as functions of the strong. This procedure is carried through in § 7. The resulting expressions for the weak coefficients will then be substituted into the strong equations (see § 8) so that in place of the infinite set of equations (6·3) in an infinite number of variables, we get four equations in the four strong coefficients. The reason why this procedure is useful lies in that we shall be able to prove that those terms in the strong equations which stem from the weak coefficients are negligible for large (but finite) M and R. The question of whether a dynamo exists then turns on whether the four strong equations have a solution.

$$\text{TABLE 2. COEFFICIENTS} \begin{pmatrix} A' \text{ or } B' & T \text{ or } S & n & m \\ & \omega & & \\ A' \text{ or} & & & \\ \text{or } B' & T \text{ or } S & n' & m' \\ RA' \text{ or } RB' & & & \end{pmatrix}.$$

(The row determines the lower line and the column the upper line. In the row (lower line) labels, m' can take on all values, including zero. In the column (upper line) labels, $m \neq 0$ unless m is explicitly put equal to zero. Coefficients with lower lines (RA' or RB' $Sn'm'$) are zero (see §5). When $n = 1$, the terms with $k = -1$ in the first and second columns are to be dropped. There are no coefficients $\begin{pmatrix} A'Sn0 \\ \omega \\ \ldots \end{pmatrix}$ because of (4·2) and (4·3).
The table remains valid if A and B are interchanged everywhere.)

upper line / lower line	$A'Tn0$	$A'Tnm \quad (m \neq 0)$	$A'Snm$
$B'Tn'm'$	$\sum_{k=\pm 1} \begin{pmatrix} AS(n+k)\,0 \\ B'Tn'm' \end{pmatrix} \begin{pmatrix} A'Tn0 \\ \omega \\ AS(n+k)\,0 \end{pmatrix}$	$\sum_{k=\pm 1} \begin{pmatrix} ASn+km \\ B'Tn'm' \end{pmatrix} \begin{pmatrix} A'Tnm \\ \omega \\ AS(n+k)\,m \end{pmatrix} + \begin{pmatrix} ATnm \\ B'Tn'm' \end{pmatrix} \begin{pmatrix} A'Tnm \\ \omega \\ ATnm \end{pmatrix}$	$\begin{pmatrix} ASnm \\ B'Tn'm' \end{pmatrix} \begin{pmatrix} A'Snm \\ \omega \\ ASnm \end{pmatrix}$
$RB'Tn'm'$	$\sum_{k=\pm 1} \begin{pmatrix} AS(n+k)\,0 \\ RB'Tn'm' \end{pmatrix} \begin{pmatrix} A'Tn0 \\ \omega \\ AS(n+k)\,0 \end{pmatrix}$	$\sum_{k=\pm 1} \begin{pmatrix} AS(n+k)\,m \\ RB'Tn'm' \end{pmatrix} \begin{pmatrix} A'Tnm \\ \omega \\ AS(n+k)\,m \end{pmatrix} + \begin{pmatrix} ATnm \\ RB'Tn'm' \end{pmatrix} \begin{pmatrix} A'Tnm \\ \omega \\ ATnm \end{pmatrix}$	$\begin{pmatrix} ASnm \\ RB'Tn'm' \end{pmatrix} \begin{pmatrix} A'Snm \\ \omega \\ ASnm \end{pmatrix}$
$B'Sn'm'$	$\sum_{k=\pm 1} \begin{pmatrix} AS(n+k)\,0 \\ B'Sn'm' \end{pmatrix} \begin{pmatrix} A'Tn0 \\ \omega \\ AS(n+k)\,0 \end{pmatrix}$	$\sum_{k=\pm 1} \begin{pmatrix} AS(n+k)\,m \\ B'Sn'm' \end{pmatrix} \begin{pmatrix} A'Tnm \\ \omega \\ AS(n+k)\,m \end{pmatrix}$	$\begin{pmatrix} ASnm \\ B'Sn'm' \end{pmatrix} \begin{pmatrix} A'Snm \\ \omega \\ ASnm \end{pmatrix}$
$RA'Tn'm'$	$\sum_{k=\pm 1} \begin{pmatrix} AS(n+k)\,0 \\ RA'Tn'm \end{pmatrix} \begin{pmatrix} A'Tn0 \\ \omega \\ AS(n+k)\,0 \end{pmatrix}$	$\sum_{k=\pm 1} \begin{pmatrix} AS(n+k)\,m \\ RA'Tn'm' \end{pmatrix} \begin{pmatrix} A'Tnm \\ \omega \\ AS(n+k)\,m \end{pmatrix} + \begin{pmatrix} ATnm \\ RA'Tn'm' \end{pmatrix} \begin{pmatrix} A'Tnm \\ \omega \\ ATnm \end{pmatrix}$	$\begin{pmatrix} ASnm \\ RA'Tn'm' \end{pmatrix} \begin{pmatrix} A'Snm \\ \omega \\ ASnm \end{pmatrix}$

7. SOLUTION OF THE WEAK EQUATIONS

In this section we shall deal with the weak equations among (6·3). An exact solution of these weak equations is not possible; it is not necessary either because all we need are proofs that a solution exists, and that the contributions of the weak a's and b's to the strong equations are negligible.

To obtain these proofs we shall proceed as follows. We shall first write down a formal successive approximation solution, assuming the strong a's and b's to be given. We shall then derive upper bounds for the weak a's and b's by applying mathematical induction to this solution. The computation of upper bounds to the terms in the strong equations stemming from the weak a's and b's is left until § 8.

Before this programme can be carried through, we need upper bounds for the coefficients in (6·3). We derive these next. From (3·12) we get

$$\left|\begin{pmatrix}\alpha n'm'\\ \beta' nm\end{pmatrix}_{\|}\right| \leqslant \frac{(n+n')!}{n!\,n'!}\left(\frac{a}{R}\right)^{n+n'+1}\left(\frac{2n+1}{2n'+1}\right)^{\frac{1}{2}}, \tag{7·1}$$

because the quantity in square brackets in (3·12) has its minimum at $m = 0$. Now for a given value of $(n+n')$, the ratio $[(n+n')!/n!\,n'!]$ has a maximum value at $n = n' = \frac{1}{2}(n+n')$ if $(n+n')$ is even, and at $n = n' \pm 1 = \frac{1}{2}(n+n' \pm 1)$ if $(n+n')$ is odd. Therefore we have for $(n+n')$ even

$$\frac{(n+n')!}{n!\,n'!} \leqslant \frac{(n+n')!}{[\frac{1}{2}(n+n')]!^2} = \frac{2^{n+n'}}{\sqrt{\pi}}\frac{[\frac{1}{2}(n+n'-1)]!}{[\frac{1}{2}(n+n')]!} < \frac{2^{n+n'}}{\sqrt{\pi}}, \tag{7·2a}$$

and for $(n+n')$ odd

$$\frac{(n+n')!}{n!\,n'!} \leqslant \frac{(n+n')!}{[\frac{1}{2}(n+n'-1)]!\,[\frac{1}{2}(n+n'+1)]!} = \frac{2^{n+n'-1}}{\sqrt{\pi}}\frac{(n+n')}{\frac{1}{2}(n+n'+1)}\frac{[\frac{1}{2}(n+n'-2)]!}{[\frac{1}{2}(n+n'-1)]!} < \frac{2^{n+n'}}{\sqrt{\pi}}. \tag{7·2b}$$

By substituting (7·2a, b) in (7·1) we get

$$\left|\begin{pmatrix}\alpha n'm'\\ \beta' nm\end{pmatrix}_{\|}\right| = O\{[n, n']\,\eta^{n+n'+1}\}, \tag{7·3}$$

where the notation $[n, n']$ means 'powers of n and n'',

$$\eta \equiv 2a/R, \tag{7·4}$$

and where we are using the O notation in the usual sense, i.e.

$$f(x) = O(g(x)) \quad \text{when} \quad x_0 < x < x_1$$

means that there exists a positive real number V independent of x (but possibly depending on x_0 and x_1) such that

$$|f| < V|g| \quad \text{when} \quad x_0 < x < x_1.$$

Here, and at all other points where we use the O notation, the intervals are

$$n \geqslant 1, \quad n' \geqslant 1.$$

For the moment (7·3) is valid for $\eta \geqslant 0$; this interval will have to be restricted later.

To get an upper bound for the $\begin{pmatrix}\alpha \ldots\\ \beta' \ldots\end{pmatrix}$ without the suffix $\|$, we have to combine (7·3) with an upper bound for the $\mathscr{D}$'s (see (3·16)). By putting $m = m'$ in (3·15), we get

$$|\mathscr{D}| \leqslant 1. \tag{7·5}$$

We now have from (3·16), (7·3) and (7·5)

$$\left|\begin{pmatrix}\alpha n'm'\\ \beta' nm\end{pmatrix}\right| = O\{[n, n']\,\eta^{n+n'+1}\}. \tag{7·6}$$

The upper bounds for $\left|\begin{pmatrix} AS \text{ or } Tn'm' \\ B'Snm \end{pmatrix}\right|$ follow immediately from (7·6) (see (3·18)).

For the upper bound of $\begin{pmatrix} ASn'm' \\ B'Tnm \end{pmatrix}$ we get from (3·20), (7·3) and (7·5)

$$\left|\begin{pmatrix} ASn'm' \\ B'Tnm \end{pmatrix}\right| = O\{[n, n']\, \eta^{n+n'}\}. \tag{7·7}$$

Table 3. Orders of magnitude of the coefficients in the field expansions (3·17), (5·2), (5·4*b*) and (5·5)

(Points in the table with a diagonal stroke correspond to coefficients which are meaningless. The O symbol and the factor $[n, n']$ have been omitted, i.e. the order of a coefficient is $O\{[n, n'] \times \text{entry in the table}\}$. The last two lines are valid only if $2\bar{R}/M < 1$, where $\bar{R} \equiv (|\mathbf{R}_A| + |\mathbf{R}_B|)$. The abbreviations are $\eta \equiv (2a/R)$, $\mu \equiv (2a/M)$, $\zeta \equiv (2\bar{R}/M)$. The table remains valid if A and B are interchanged everywhere.)

lower line \ upper line	$BSn'm'$	$BTn'm'$	$C'Tn'm'$
$RB'Tnm$	$\eta^{n+n'+1}$	0	0
$RA'Tnm$	$\eta^{n+n'}$	$\eta^{n+n'+1}$	0 if $n' < n$, $\mu^{n+1}\zeta^{n'-n}$ if $n' \geqslant n$
$CTnm$	0 if $n < n'$, $\mu^{n'-1}\zeta^{n-n'+1}$ if $n \geqslant n'$	0 if $n < n'$, $\mu^{n'}\zeta^{n-n'}$ if $n \geqslant n'$	╲
$A'Tnm$	$\mu^{n+n'}\zeta^{\|n-n'\|+1}$	$\mu^{n+n'+1}\zeta^{\|n-n'\|}$	╲
$A'Snm$	$\mu^{n+n'}\zeta^{\|n-n'\|+1}$	$\mu^{n+n'+1}\zeta^{\|n-n'\|}$	╲

To get an upper bound for $\begin{pmatrix} ATn'm' \\ B'Tnm \end{pmatrix}$ from (3·27), we need an estimate of the integral appearing there. By using Schwarz's inequality and the normalization of the Y_{nm}, we find

$$\left|\int \mathrm{d}\Omega_A\, Y^*_{n'm'}(\theta_A, \lambda_A)\, (\mathbf{R}\,.\,\nabla\alpha_{(n'+1)m''})_{r_A=a}\right|^2 \leqslant \int \mathrm{d}\Omega_A (\mathbf{R}\,.\,\nabla\alpha^*_{(n'+1)m''})\,(\mathbf{R}\,.\,\nabla\alpha_{(n'+1)m''})$$
$$\leqslant \frac{R^2}{a^2}\int_{r_A=a} \mathrm{d}S(\nabla\alpha^*_{(n'+1)m''}\,.\,\nabla\alpha_{(n'+1)m''}) = \frac{R^2}{a^2}\{(n'+1)^2 + (n'+1)\,(n'+2)\}. \tag{7·8}$$

(The first term in (7·8) comes from the radial, and the second from the angular derivatives.) From (3·27), (7·6) and (7·8) we find

$$\left|\begin{pmatrix} ATn'm' \\ B'Tnm \end{pmatrix}\right| = O\{[n, n']\, \eta^{n+n'+1}\}. \tag{7·9}$$

Estimates for the coefficients in (5·2) and (5·4*b*) can be obtained by similar arguments. The results, together with those we have just obtained, are given in the first three lines of table 3. In this table

$$\mu \equiv \frac{2a}{M}, \quad \zeta \equiv \frac{2\bar{R}}{M}, \tag{7·10a}$$

where

$$\bar{R} \equiv |\mathbf{R}_A| + |\mathbf{R}_B|. \tag{7·10b}$$

The computation of upper bounds for the coefficients $\begin{pmatrix} A \text{ or } B \dots \\ RA' \dots \end{pmatrix}$ from (5·5) requires the summation of an infinite series. To do this, we shall use the following lemma:

If $|n_0| \geq 1$, $S_0 \geq S \geq 0$ *for some* S_0, *and* $|x| \leq |x_0| \leq 1$ *for some* x_0, *then*

$$\sum_{n=n_0}^{\infty} n^S x^n = O(n_0^S x^{n_0}). \tag{7·11}$$

This lemma states that the series in (7·11) has an upper bound whose form is that of the first term; we shall therefore call (7·11) the 'first-term lemma'.

To prove the lemma (7·11), we have, under the conditions stated

$$\left|\sum_{n=n_0}^{\infty} n^S x^n\right| \leq |n_0^S| \, |x^{n_0}| \sum_{t=0}^{\infty} |x_0^t| \left|\left(\frac{n_0+t}{n_0}\right)^S\right|;$$

but

$$\left|\left(\frac{n_0+t}{n_0}\right)\right| \leq \left(1+\left|\frac{t}{n_0}\right|\right) \leq (1+|t|)$$

and

$$\left|\left(\frac{n_0+t}{n_0}\right)\right|^S \leq (1+|t|)^{S_0},$$

so that

$$\left|\sum_{n=n_0}^{\infty} n^S x^n\right| \leq |n_0^S| \, |x^{n_0}| \sum_{t=0}^{\infty} |x_0^t| \, (1+t)^{S_0}.$$

Since $|x_0| < 1$, the series on the right converges and represents a function of S_0 and x_0 which is independent of n_0, S and x; this proves the lemma. In our applications the condition $|n_0| \geq 1$ will be satisfied, and all powers of n that enter are finite so that S_0 exists. Thus the only condition we have to watch is that on x.

We can now obtain an upper bound for the coefficients $\begin{pmatrix} B \dots \\ RA' \dots \end{pmatrix}$ from (5·5) by replacing the terms on the right by their moduli, substituting from the first three lines of table 3, and applying the first-term lemma, which is valid provided that

$$\zeta \leq \zeta_0 < 1 \quad \text{for some } \zeta_0. \tag{7·12a}$$

This condition is a physical restriction which is not required by the geometry of our model (see figure 1), but arose from our method of constructing upper bounds to $\begin{pmatrix} A \text{ or } B \dots \\ C \dots \end{pmatrix}$ and $\begin{pmatrix} C' \dots \\ A' \text{ or } B' \dots \end{pmatrix}$. An analogous argument gives $\begin{pmatrix} A \dots \\ RA' \dots \end{pmatrix}$. The results are given in the last two lines of table 3.

We next compute upper bounds for the complete coefficients in the weak equations given in table 2. To do this, we have to combine the upper bounds in table 3 with the induction coefficients (4·2), (4·3), (4·8), (4·9) and (4·15) in accordance with table 2. In writing these upper bounds, we shall assume that

$$0 \leq \eta \leq \eta_0 < 1 \quad \text{for some } \eta_0 \tag{7·12b}$$

and

$$0 \leq \mu \leq \mu_0 < 1 \quad \text{for some } \mu_0. \tag{7·12c}$$

Both these conditions are consequences of the geometry of our model (see figure 1). In the sums $\sum_{k=\pm 1}$ in table 2, only the term $k = +1$ occurs when $n = 1$ in the upper line of the coefficient

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

because there is no field ($ASn=0m$). For values of $n>1$, both the terms $k=\pm 1$ occur, but the ratio of the term with $k=+1$ to that with $k=-1$ contains, except for a factor $[n, n']$, the factor η^2 in $\begin{pmatrix} A' \dots \\ B \dots \end{pmatrix}$, and the factor $\mu^2 \zeta^{|n-n'+1|-|n-n'-1|}$ in $\begin{pmatrix} A \text{ or } B \dots \\ RB' \dots \end{pmatrix}$. The largest value of $\mu^2 \zeta^{|n-n'+1|-|n-n'-1|}$ occurring is $(\mu/\zeta)^2$ (remember $\zeta<1$), and this is

$$\left(\frac{\mu}{\zeta}\right)^2 = \left(\frac{2a}{M} \times \frac{M}{2R}\right)^2 = \left(\frac{2a}{R} \times \frac{R}{2R}\right)^2 = \eta^2 \left[\frac{|\mathbf{R}_A - \mathbf{R}_B|}{2(|\mathbf{R}_A| + |\mathbf{R}_B|)}\right]^2 < \eta^2. \tag{7·13}$$

Since $\eta^2<1$, we need take only the terms with $k=-1$ when computing the orders of terms with $n>1$ in table 2.

Table 4. Order of coefficients in equation (6·3)

(The symbols O and factors $[n, n']$ have been omitted, i.e. any particular coefficient is

$O\{[n, n'] \times \text{entry in the table}\}$.

The table remains valid if A and B are interchanged everywhere.)

upper line / lower line	$A'Tn0$ ($n=1$)	$A'Tn0$ ($n>1$)	$A'Tnm$ ($m\neq 0$, $n=1$)	$A'Tnm$ ($m\neq 0$, $n>1$)	$A'Sn0$	$A'Snm$ ($m\neq 0$)
$B'Tn'm'$	$\omega a^2\sigma \times \eta^{n'+2}$	$\omega a^2\sigma \times \eta^{n+n'-1}$	$\eta^{n'+2}$	$\eta^{n+n'-1}$	0	$\eta^{n+n'}$
$B'Sn'm'$	$\omega a^2\sigma \times \eta^{n'+3}$	$\omega a^2\sigma \times \eta^{n+n'}$	$\eta^{n'+3}$	$\eta^{n+n'}$	0	$\eta^{n+n'+1}$
$RB'Tn'm'$ or $RA'Tn'm'$	$\omega a^2\sigma \times \mu^3\zeta^2$ if $n'=1$; $\omega a^2\sigma \times \mu^{n'+2}\zeta^{n'-1}$ if $n'>1$	$\omega a^2\sigma \times \mu^{n+n'-1} \times \zeta^{\lvert n'-n+1\rvert+1}$	μ^3 if $n'=1$; $\mu^{n'+2}\zeta^{n'-1}$ if $n'>1$	$\mu^{n+n'-1} \times \zeta^{\lvert n-n'-1\rvert+1}$	0	$\mu^{n+n'} \times \zeta^{\lvert n-n'\rvert+1}$

The upper bounds of the terms in table 2, and in particular of the coefficients of the weak equations, are given in table 4. All the entries in the first and second columns have a factor $\omega a^2\sigma$; these factors will come to play the important role of making dynamo action possible by compensating for the factors η which represent the fall-off of the induced fields with distance. In the coefficients with upper line $A'Tnm$ ($m\neq 0$) in the third and fourth columns, the orders are determined as follows: in $\begin{pmatrix} A'T\dots \\ \omega \\ B'T\dots \end{pmatrix}$, the second term, containing $\begin{pmatrix} AT\dots \\ B'T\dots \end{pmatrix}$, is smaller than the first, containing $\begin{pmatrix} AS\dots \\ B'T\dots \end{pmatrix}$, by a factor $O(\eta^2)$ for $n>1$, and can be ignored, while for $n=1$ the two are of the same order in η. In the coefficients $\begin{pmatrix} AT\dots \\ \omega \\ RA' \text{ or } RB'\dots \end{pmatrix}$, the second term is smaller than the first for $n>1$ by at least a factor $(\mu/\zeta)^2$, which is less than η^2 by (7·13) and can be ignored; for $n=1$, the two terms in $\begin{pmatrix} AT\dots \\ \omega \\ RA' \text{ or } RB'\dots \end{pmatrix}$ are of the same order when $n'>1$, while the first is smaller by a factor ζ^2 for $n'=1$, when the second term becomes predominant.

With the upper bounds in table 4, we can now start on the programme outlined at the beginning of this section for the solution of the weak equations (6·3). We repeat that all that is required is a proof that a solution exists, and an upper bound for those terms in the strong equations which contain weak a's and b's.

We start by solving the weak equations (6·3) by successive approximations, i.e. we write

$$a_{{}^{S}_{T}nm} = \sum_{i=1}^{\infty} a^{(i)}_{{}^{S}_{T}nm}, \quad b_{{}^{S}_{T}nm} = \sum_{i=1}^{\infty} b^{(i)}_{{}^{S}_{T}nm}, \tag{7·14}$$

where $(n, m) \neq (1, 0)$ or $(2, 0)$ for the a_{Tnm} and b_{Tnm}, and where

$$a^{(i)}_{{}^{S}_{T}nm} = \sum_{n'=1}^{\infty}{}' \sum_{m'=-n'}^{n'}{}' \left\{ b^{(i-1)}_{Sn'm'} \begin{pmatrix} A' \, {}^{S}_{T}nm \\ \omega \\ B'Sn'm' \end{pmatrix} + b^{(i-1)}_{Tn'm'} \left[\begin{pmatrix} A' \, {}^{S}_{T}nm \\ \omega \\ B'Tn'm' \end{pmatrix} + \begin{pmatrix} A' \, {}^{S}_{T}nm \\ \omega \\ RB'Tn'm' \end{pmatrix} \right] + a^{(i-1)}_{Tn'm'} \begin{pmatrix} A' \, {}^{S}_{T}nm \\ \omega \\ RA'Tn'm' \end{pmatrix} \right\}, \tag{7·15}$$

with a similar equation for the $b^{(i)}$. The primes on the Σ signs imply that the strong a's and b's are excluded from the summation, except for $i = 1$ when only the strong and no weak a's and b's appear on the right-hand side.

We now construct upper bounds for the $a^{(i)}$ and $b^{(i)}$, and prove that the series (7·14) converge. Let us suppose that there exist positive numbers $t^{(i)}_{T0}$, $t^{(i)}_{T1}$ and $t^{(i)}_{TS}$ such that

$$|a^{(i)}_{Tn0}|, \quad |b^{(i)}_{Tn0}| < t^{(i)}_{T0} \quad \text{for} \quad n \geq 3, \tag{7·16a}$$

$$|a^{(i)}_{T1m}|, \quad |b^{(i)}_{T1m}| < t^{(i)}_{T1} \quad \text{for} \quad m \neq 0, \tag{7·16b}$$

$$|a^{(i)}_{Tnm}|, \quad |b^{(i)}_{Tnm}| < t^{(i)}_{TS} \quad \text{for} \quad n \geq 2,\ m \neq 0, \tag{7·16c}$$

$$|a^{(i)}_{Snm}|, \quad |b^{(i)}_{Snm}| < t^{(i)}_{TS}. \tag{7·16d}$$

Note that the same quantity $t^{(i)}_{TS}$ appears on the right of both equations (7·16c) and (7·16d).†

† The reasons for our grouping of different a's and b's under common upper bounds in (7·16) are as follows. Let us for the moment ignore the effect of the conductor surface, i.e. put $M = \infty$. The separation of the a_{Tnm} and b_{Tnm} with $m = 0$ from the rest suggests itself because of the appearance of the amplification factors $\omega a^2 \sigma$ in columns 1 and 2 of table 4, and their absence from columns 3 and 4; no similar separation of the a_{Snm} and b_{Snm} with $m = 0$ is necessary because the zeros in column 5 of table 4 show that the corresponding fields are not excited.

The separation of the a_{Tnm} and b_{Tnm} with $n = 1$ $(m \neq 0)$ from those with $n > 1$ $(m \neq 0)$ suggests itself because the former are excited less easily than the lowest in n of the latter by a factor η (compare columns 3 and 4 of table 4).

Finally, we group together the a_{Tnm} and b_{Tnm} with $n > 1$ $(m \neq 0)$ with the a_{Snm} and b_{Snm} because the lowest member of each group ($n = 2$ and 1, respectively) contains the same power of η both in the factor responsible for its excitation (compare columns 4 and 6) and in the fields it produces (compare lines 1 and 2).

If we now consider the effect of a finite M, then we have to look at line 3 of table 4. If we ignore the factors ζ, which by (7·12) may be as large, but no larger, than unity, then line 3 differs from line 1 only in that the former has μ where the latter has η. Since the upper bound of (μ/η) is $\frac{1}{2}$, because

$$\frac{\mu}{\eta} = \frac{R}{M} < \frac{\overline{R}}{M} = \tfrac{1}{2}\zeta < \tfrac{1}{2}, \tag{7·17}$$

the reflexion terms on the right of (6·3) are of the same order of importance as those not involving reflexion, at least for low values of n. This fact suggests the grouping together of a's and b's under common upper bounds.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

We next obtain a recurrence relation between the $t^{(i)}$ and $t^{(i-1)}$. By replacing the terms on the right of (7·15) by their moduli, and using (7·16), we get, for $i>1$,

$$\begin{aligned}|a^{(i)}_{\substack{S\\T}nm}| < t^{(i-1)}_{T0} \sum_{n'=3}^{\infty} \left[\left|\begin{pmatrix} A' & \overset{S}{T}nm \\ & \omega \\ B' & Tn'0 \end{pmatrix}\right| + \left|\begin{pmatrix} A' & \overset{S}{T}nm \\ & \omega \\ RB' & Tn'0 \end{pmatrix}\right| + \left|\begin{pmatrix} A' & \overset{S}{T}nm \\ & \omega \\ RA' & Tn'0 \end{pmatrix}\right|\right] \\ + t^{(i-1)}_{T1} \sum_{m'=\pm 1} \left[\left|\begin{pmatrix} A' & \overset{S}{T}nm \\ & \omega \\ B' & T1m' \end{pmatrix}\right| + \left|\begin{pmatrix} A' & \overset{S}{T}nm \\ & \omega \\ RB' & T1m' \end{pmatrix}\right| + \left|\begin{pmatrix} A' & \overset{S}{T}nm \\ & \omega \\ RA' & T1m' \end{pmatrix}\right|\right] \\ + t^{(i-1)}_{TS} \left\{ \sum_{n'=2}^{\infty} \sum_{\substack{m'=-n' \\ m'\neq 0}}^{n'} \left[\left|\begin{pmatrix} A' & \overset{S}{T}nm \\ & \omega \\ B' & Tn'm' \end{pmatrix}\right| + \left|\begin{pmatrix} A' & \overset{S}{T}nm \\ & \omega \\ RB' & Tn'm' \end{pmatrix}\right| + \left|\begin{pmatrix} A' & \overset{S}{T}nm \\ & \omega \\ RA' & Tn'm' \end{pmatrix}\right|\right] \right. \\ \left. + \sum_{n'=1}^{\infty} \sum_{\substack{m'=-n' \\ m'\neq 0}}^{n'} \left|\begin{pmatrix} A' & \overset{S}{T}nm \\ & \omega \\ B' & Sn'm' \end{pmatrix}\right| \right\}. \end{aligned} \tag{7·18}$$

The orders of the sums in (7·18) can be obtained by substituting from table 4 and applying the first-term lemma. This calculation is simplified a little if we note that line 3 of table 4 remains valid if we replace ζ by unity (we then overestimate the coefficients whose orders appear in that line). Line 3 then differs from line 1 only in the appearance of μ in place of η, and since $\mu<\frac{1}{2}\eta$ (by (7·17)) it follows that the orders of the second and third terms on the right in the first three lines of (7·18) do not exceed that of the first term in their respective line. Therefore, when computing orders, we need consider only the first term in each line in (7·18). One of the results obtained is

$$|a^{(i)}_{Tn0}| = \omega\sigma a^2\{O([n]\,\eta^{n+2})\,t^{(i-1)}_{T0} + O([n]\,\eta^{n+1})\,t^{(i-1)}_{TS} + O([n]\,\eta^{n})\,t^{(i-1)}_{T1}\}; \tag{7·19}$$

a similar upper bound holds for $|b^{(i)}_{Tn0}|$.

To simplify the expression (7·19), we shall introduce a quantity χ defined by

$$\frac{1}{\chi} \equiv \frac{\omega\sigma a^2\eta^3}{40}, \tag{7·20a}$$

and assume that there exists a lower bound χ_0 such that

$$|\chi| \geqslant \chi_0 > 0. \tag{7·20b}$$

For the moment the limit χ_0 can be taken to be a pure number independent of ω, σ, a and η; its magnitude will be defined in § 8. (The factor 1/40 will be convenient in § 8.) The usefulness of equations (7·20a) and (7·20b) is due to their allowing us to write

$$\omega\sigma a^2\eta^3 = O(1) \quad \text{when} \quad 0 \leqslant \eta < 1. \tag{7·20c}$$

The physical significance of the restriction (7·20c) is as follows: we have already seen (in § 6) that a dynamo based on the strong a's and b's probably requires that ω be proportional to R^3, or, in other words, that $\omega\eta^3$ be constant as R varies, whereas dynamos based on what we prefer to regard as weak a's and b's require a more rapid variation of ω with R.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Therefore, in introducing the restriction (7·20c) into the discussion of the convergence of the solution of the weak equations, we are trying to use the fact that an ω which is large enough to give regeneration through the strong a's and b's, will in general be too small for the weak a's and b's to be of any serious importance.

To get a recurrence relation for the $t^{(i)}$ from (7·19), we use (7·20c) in (7·19) and get

$$|a^{(i)}_{Tn0}| = O([n]\,\eta^{n-1})\,t^{(i-1)}_{T0} + O([n]\,\eta^{n-2})\,t^{(i-1)}_{TS} + O([n]\,\eta^{n-3})\,t^{(i-1)}_{T1}. \tag{7·21 a}$$

We can simplify this result by noting that if $n \geqslant n_0$, then

$$|\{[n]\,\eta^n\}| = \eta^{n_0} \times |\{[n]\,\eta^{n-n_0}\}| = O([n_0]\,\eta^{n_0}) \tag{7·22}$$

because the factor $|\{\ \}|$ is bounded by a number depending only on η_0 (see (7·12b)), n_0, and the highest index occurring in $[n]$, but independent of η and n. If we use (7·22) in (7·21a), then we get a common upper bound for all the $a^{(i)}_{Tn0}$ with $n = n_0, n_0+1, \ldots$. Putting $n_0 = 3$, we may write

$$t^{(i)}_{T0} = O(\eta^2)\,t^{(i-1)}_{T0} + O(\eta)\,t^{(i-1)}_{TS} + O(1)\,t^{(i-1)}_{T1} \tag{7·21 b}$$

when $$0 \leqslant \eta \leqslant \eta_0 < 1.$$

Results, similar to (7·21b) can be obtained from (7·18) for $t^{(i)}_{TS}$ and $t^{(i)}_{T1}$. These results can be grouped together in the matrix form

$$[t^{(i)}] < VU[t^{(i-1)}], \tag{7·23}$$

where V is some finite positive number, while

$$[t^{(i)}] \equiv \begin{bmatrix} t^{(i)}_{T0} \\ t^{(i)}_{TS} \\ t^{(i)}_{T1} \end{bmatrix} \quad \text{and} \quad U \equiv \begin{bmatrix} \eta^2 & \eta & 1 \\ \eta^4 & \eta^3 & \eta^2 \\ \eta^5 & \eta^4 & \eta^3 \end{bmatrix}. \tag{7·24}$$

The inequality (7·23) is our desired recurrence relation for the $t^{(i)}$. By its construction we have demonstrated that if there exists a set of upper bounds $t^{(i-1)}$, then there also exists a set $t^{(i)}$.

We next compute $[t^{(1)}]$, and thereby show that all the $[t^{(i)}]$ are finite. The weak a's and b's to first approximation (i.e. $a^{(1)}$, $b^{(1)}$) are obtained from (7·15) by retaining on the right only terms containing strong a's and b's. Upper bounds are obtained by replacing all terms in the sums by their moduli, substituting from table 4, and using (7·20c) and (7·22). One finds

$$[t^{(1)}] \equiv \begin{bmatrix} t^{(1)}_{T0} \\ t^{(1)}_{TS} \\ t^{(1)}_{T1} \end{bmatrix} = O\begin{bmatrix} \eta \\ \eta^3 \\ \eta^4 \end{bmatrix} (|\bar{a}_{T10}| + |\bar{b}_{T10}| + |a_{T20}| + |b_{T20}|), \tag{7·25}$$

where $$\bar{a}_{T10} \equiv 2\left(\frac{3}{5}\right)^{\frac{1}{2}} \frac{1}{\eta}\, a_{T10}, \quad \bar{b}_{T10} \equiv 2\left(\frac{3}{5}\right)^{\frac{1}{2}} \frac{1}{\eta}\, b_{T10}. \tag{7·26}$$

The renormalization (7·26) is convenient because it avoids odd factors η and other numerical factors, both in (7·25) and at other places later.

We next use (7·23) to show that the successive approximation solution (7·14) converges. By repeated application of (7·23), we find

$$[t^{(i)}] < V^{(i-1)}U^{(i-1)}[t^{(1)}]. \tag{7·27}$$

By tåking moduli on both sides of (7·14), and using (7·16) and (7·27), we have then

$$\begin{bmatrix} |a_{Tn0}| & (n \geqslant 3) \\ |a_{Tnm}| & (m \neq 0,\ n \geqslant 2) \quad \text{or} \quad |a_{Snm}| \\ |a_{T1m}| & (m \neq 0) \end{bmatrix} < \sum_{i=1}^{\infty} [t^{(i)}] < [t^{(1)}] + \left(\sum_{i=1}^{\infty} V^i U^i\right) [t^{(1)}]. \tag{7·28}$$

To compute the sum of matrix products in (7·28), we try to find a matrix T which has the property

$$T^{-1}UT = U_D, \tag{7·29}$$

where U_D is diagonal. With the aid of T, one has, by inserting a factor TT^{-1} between successive U's in (7·28)

$$\sum_{i=1}^{\infty} V^i U^i = T\left[\sum_{i=1}^{\infty} V^i (T^{-1}UT)^i\right] T^{-1} = T\left(\sum_{i=1}^{\infty} V^i U_D^i\right) T^{-1}. \tag{7·30}$$

The sum can now be calculated provided that $(|V| \times \text{any element of } U_D) < 1$.

A matrix T which satisfies (7·29) is

$$T = (1+2\eta)^{-\frac{1}{2}} \begin{bmatrix} 1 & -1 & 0 \\ \eta^2 & 0 & -\eta \\ \eta^3 & \eta^2 & \eta^2 \end{bmatrix}, \quad T^{-1} = (1+2\eta)^{-\frac{1}{2}} \begin{bmatrix} 1 & \dfrac{1}{\eta} & \dfrac{1}{\eta^2} \\ -2\eta & \dfrac{1}{\eta} & \dfrac{1}{\eta^2} \\ \eta & -\left(\dfrac{1+\eta}{\eta}\right) & \dfrac{1}{\eta} \end{bmatrix}, \tag{7·31}$$

which gives

$$U_D = \begin{bmatrix} \eta^2(1+2\eta) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{7·32}$$

(The columns of T are the eigenvectors of U; the rows of T^{-1} are the eigenvectors of the transpose of U; and the elements of U_D are the eigenvalues of U.)

Using (7·32) in (7·30), we get

$$\sum_{i=1}^{\infty} V^i U^i = T \begin{bmatrix} \dfrac{V\eta^2(1+2\eta)}{1 - V\eta^2(1+2\eta)} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} T^{-1} \tag{7·33}$$

$$= \frac{V}{1 - V\eta^2(1+2\eta)} T U_D T^{-1} = \frac{V}{1 - V\eta^2(1+2\eta)} U,$$

provided that

$$V\eta^2(1+2\eta) < 1. \tag{7·34}$$

The inequality (7·34) is the condition that the matrix sum in (7·28) should converge, and therefore its satisfaction is a sufficient condition for the convergence of our successive approximation solution (7·14) and (7·15).

From (7·25), (7·28), (7·33) and (7·24), we finally get the following upper bounds for the complete weak a's and b's:

$$\begin{bmatrix} |a_{Tn0}| & (n \geqslant 3) \\ |a_{Tnm}| & (m \neq 0,\ n \geqslant 2) \quad \text{or} \quad |a_{Snm}| \\ |a_{T1m}| & (m \neq 0) \end{bmatrix} = O\begin{bmatrix} \eta \\ \eta^3 \\ \eta^4 \end{bmatrix} (|\bar{a}_{T10}| + |\bar{b}_{T10}| + |a_{T20}| + |b_{T20}|); \tag{7·35}$$

there are similar upper bounds for the b's.

The only restrictive conditions, other than those due to the geometry of the model, to which this result is subject are (7·12*a*) and (7·34). We should emphasize that all we know about V in (7·34) is that it is a finite positive number independent of all variables.

For later reference, we next prove that the weak a's and b's are continuous functions of ω. The series (7·15) for $a^{(i)}$ is uniformly convergent with respect to ω, because our derivation of (7·19) from (7·18), and similar arguments for $a^{(i)}_{Tnm}$ and $a^{(i)}_{Snm}$ show that Weierstrass's M-test can be satisfied. Therefore the $a^{(i)}$ and $b^{(i)}$ are continuous functions of ω if the $a^{(i-1)}$ and $b^{(i-1)}$ are. But the $a^{(1)}$ and $b^{(1)}$ are manifestly continuous in ω; therefore the $a^{(i)}$ and $b^{(i)}$ for all i are continuous in ω. The sums (7·14) for the complete weak a's and b's converge uniformly with respect to ω because Weierstrass's M-test is satisfied (see (7·16), (7·27), (7·33) and (7·34)). Therefore the weak a's and b's are continuous, because they are given by a uniformly convergent series of continuous functions.

8. Solution of the strong equations

We next take the strong equations (6·3), and express them entirely in terms of the strong a's and b's by using the formula (7·35). We then show that our model can act as a dynamo by proving that under certain conditions these equations can have a solution for real values of the parameters.

After the weak a's and b's have been expressed in terms of the strong, the strong equations (6·3) can be written in the form

$$[a] = \left\{\begin{bmatrix} A' \\ \omega \\ B' \end{bmatrix} + \begin{bmatrix} A' \\ \omega \\ RB' \end{bmatrix} + \begin{bmatrix} A' \\ \omega \\ WB' \end{bmatrix}\right\}[b] + \left\{\begin{bmatrix} A' \\ \omega \\ RA' \end{bmatrix} + \begin{bmatrix} A' \\ \omega \\ WA' \end{bmatrix}\right\}[a], \tag{8·1a}$$

$$[b] = \left\{\begin{bmatrix} B' \\ \omega \\ A' \end{bmatrix} + \begin{bmatrix} B' \\ \omega \\ RA' \end{bmatrix} + \begin{bmatrix} B' \\ \omega \\ WA' \end{bmatrix}\right\}[a] + \left\{\begin{bmatrix} B' \\ \omega \\ RB' \end{bmatrix} + \begin{bmatrix} B' \\ \omega \\ WB' \end{bmatrix}\right\}[b], \tag{8·1b}$$

where (see (7·26))

$$[a] \equiv \begin{bmatrix} \bar{a}_{T10} \\ a_{T20} \end{bmatrix}, \quad [b] \equiv \begin{bmatrix} \bar{b}_{T10} \\ b_{T20} \end{bmatrix}. \tag{8·2}$$

The symbols $\begin{bmatrix} \cdots \\ \omega \\ \cdots \end{bmatrix}$ denote 2×2 matrices. $\begin{bmatrix} A' \\ \omega \\ B' \end{bmatrix}$ represents the major part of the feed from rotator B to rotator A; this would be present even if there were no surface reflexion (i.e. $M\to\infty$) and if the weak a's and b's were ignored. $\begin{bmatrix} A' \\ \omega \\ RB' \end{bmatrix}$ represents the feed from B to A through the field RB' reflected from the conductor surface, again ignoring the weak a's and b's, and similarly $\begin{bmatrix} A' \\ \omega \\ RA' \end{bmatrix}$ represents the feed of the strong a's to themselves via surface reflexion. Finally, $\begin{bmatrix} A' \\ \omega \\ WB' \end{bmatrix}$ and $\begin{bmatrix} A' \\ \omega \\ WA' \end{bmatrix}$ represent the feed from the strong b's and a's respectively to the strong a's through the intermediary of the weak a's and b's.

The matrices $\begin{bmatrix} A' \\ \omega \\ B' \end{bmatrix}$ and $\begin{bmatrix} B' \\ \omega \\ A' \end{bmatrix}$ can be written down from table 2. The coefficients can then be expressed in terms of the parameters of the model, either in terms of the upper bounds from table 4, or exactly by means of (3·12), (3·13), (3·20), (3·22) and (4·3). One obtains

$$\begin{bmatrix} A' \\ \omega \\ B' \end{bmatrix} = \frac{1}{\chi}\left\{\sin\Theta_A \sin\Theta_B \sin\Lambda_{AB}\begin{bmatrix}\cos\Theta_A & -4\cos\Theta_A\cos\Theta_B + \sin\Theta_A\sin\Theta_B\cos\Lambda_{AB} \\ -\frac{1}{3} & \cos\Theta_B\end{bmatrix} + O(\eta^2)\right\} \quad (8\cdot 3a)$$

$$\equiv \frac{1}{\chi}\{P+O(\eta^2)\};$$

$$\begin{bmatrix} B' \\ \omega \\ A' \end{bmatrix} = \frac{1}{\chi}\left\{\sin\Theta_A \sin\Theta_B \sin\Lambda_{AB}\begin{bmatrix}-\cos\Theta_B & -4\cos\Theta_A\cos\Theta_B + \sin\Theta_A\sin\Theta_B\cos\Lambda_{AB} \\ -\frac{1}{3} & -\cos\Theta_A\end{bmatrix} + O(\eta^2)\right\} \quad (8\cdot 3b)$$

$$\equiv \frac{1}{\chi}\{Q+O(\eta^2)\};$$

where (as in § 7)

$$\frac{1}{\chi} \equiv \frac{\omega\sigma a^2\eta^3}{40}, \quad \eta \equiv \frac{2a}{R}, \quad (8\cdot 4)$$

and where

$$\Lambda_{AB} \equiv \Lambda_A - \Lambda_B. \quad (8\cdot 5)$$

$O(\eta^2)$ denotes a 2×2 matrix whose elements are $O(\eta^2)$. The angles Θ_A, Λ_A, Θ_B and Λ_B, which were introduced in § 3 (before (3·14)), are the Eulerian angles of the rotations which send $\hat{\boldsymbol{\omega}}_A$ and $\hat{\boldsymbol{\omega}}_B$ from being parallel to $\mathbf{R} = -\mathbf{R}_A + \mathbf{R}_B$ into their actual positions. The matrices P and Q defined by (8·3) will be convenient later.

The matrices (8·3) are very important because they describe the major part of the interaction between the two rotators. (8·3*b*) is obtained from (8·3*a*) by exchanging (Θ_A, Λ_A) with (Θ_B, Λ_B), and some changes of sign; there is a difference of sign from (8·5), and this is cancelled by another from (3·19) (remember that $\mathbf{R} = \mathbf{R}_B - \mathbf{R}_A$); finally there is a difference of sign from (3·13) which affects only some of the matrix elements.

Of the elements of the matrices $\begin{bmatrix} A' \text{ or } B' \\ \omega \\ RA' \text{ or } RB' \end{bmatrix}$, some can be computed to lowest order in ζ from (5·19), table 2 and (4·3).† The remainder can be given upper bounds from table 4. Remembering that

$$\frac{\mu}{\eta} = \left(\frac{R}{2\bar{R}}\right)\zeta = O(\zeta),$$

one finds that

$$\begin{bmatrix} A' \\ \omega \\ RA' \end{bmatrix} = \frac{1}{\chi}O(\zeta^5), \quad \begin{bmatrix} B' \\ \omega \\ RB' \end{bmatrix} = \frac{1}{\chi}O(\zeta^5), \quad (8\cdot 6a)$$

$$\begin{bmatrix} A' \\ \omega \\ RB' \end{bmatrix} = \frac{1}{\chi}\left\{\begin{bmatrix} 0 & 0 \\ p_0 & 0\end{bmatrix} + O(\zeta^5)\right\}, \quad (8\cdot 6b)$$

where

$$p_0 \equiv +\frac{1}{24}\frac{R^2}{\bar{R}^3}\mathbf{R}_A\cdot(\hat{\boldsymbol{\omega}}_A \wedge \hat{\boldsymbol{\omega}}_B)\,\zeta^3,$$

† (5·19) actually gives the terms of lowest n'' in the series in (5·5) for $n = n'$, and S in the upper line; by substituting in (5·5) from table 3 and applying the first-term lemma, one finds that the sum of the higher terms in (5·5) is smaller by a factor $O(\zeta^2)$.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

and
$$\begin{bmatrix} B' \\ \omega \\ RA' \end{bmatrix} = \frac{1}{\chi}\left\{\begin{bmatrix} 0 & 0 \\ q_0 & 0 \end{bmatrix} + O(\zeta^5)\right\}, \tag{8·6c}$$

where
$$q_0 \equiv +\frac{1}{24}\frac{R^2}{\bar{R}^3}\mathbf{R}_B.(\hat{\boldsymbol{\omega}}_B \wedge \hat{\boldsymbol{\omega}}_A)\,\zeta^3.$$

Equation (8·6*a*) follows from table 4 and the remarks after (5·19). For later reference, we note that the terms p_0 and q_0 in (8·6*b*) and (8·6*c*) are larger than the O terms by factors $O(\zeta^{-2})$.

The terms in (8·1) containing matrices with W represent the expression

$$\begin{bmatrix} A' \\ \omega \\ WB' \end{bmatrix}[b] + \begin{bmatrix} A' \\ \omega \\ WA' \end{bmatrix}[a] \equiv \begin{bmatrix} \frac{2}{\eta}\left(\frac{3}{5}\right)^{\frac{1}{2}} & 0 \\ 0 & 1 \end{bmatrix}$$
$$\times \sum_{n=1}^{\infty}{}' \sum_{m=-n}^{n}{}' \left\{ b_{Snm}\begin{bmatrix} \begin{pmatrix} A'T10 \\ \omega \\ B'Snm \end{pmatrix} \\ \begin{pmatrix} A'T20 \\ \omega \\ B'Snm \end{pmatrix} \end{bmatrix} + b_{Tnm}\begin{bmatrix} \begin{pmatrix} A'T10 \\ \omega \\ B'Tnm \end{pmatrix} + \begin{pmatrix} A'T10 \\ \omega \\ RB'Tnm \end{pmatrix} \\ \begin{pmatrix} A'T20 \\ \omega \\ B'Tnm \end{pmatrix} + \begin{pmatrix} A'T20 \\ \omega \\ RB'Tnm \end{pmatrix} \end{bmatrix} + a_{Tnm}\begin{bmatrix} \begin{pmatrix} A'T10 \\ \omega \\ RA'Tnm \end{pmatrix} \\ \begin{pmatrix} A'T20 \\ \omega \\ RA'Tnm \end{pmatrix} \end{bmatrix} \right\}, \tag{8·7}$$

and a similar set of terms with (A, B) and (a, b) interchanged. The primes on the Σ signs signify that the strong a's and b's are to be omitted in the summation. Upper bounds for the series in (8·7) can be computed by using (7·35), table 4 and the first-term lemma (7·11). One obtains (remember the renormalization (7·26))

$$\begin{bmatrix} A' \\ \omega \\ WB' \end{bmatrix}[b] + \begin{bmatrix} A' \\ \omega \\ WA' \end{bmatrix}[a] = \frac{1}{\chi}O(\eta^2)\,(|\bar{a}_{T10}| + |\bar{b}_{T10}| + |a_{T20}| + |b_{T20}|). \tag{8·8}$$

Since all the equations (6·3) are linear, we can drop the modulus signs in (8·8) and write

$$\begin{bmatrix} A' \\ \omega \\ WB' \end{bmatrix} = \frac{1}{\chi}O(\eta^2) \quad \text{and} \quad \begin{bmatrix} B' \\ \omega \\ WA' \end{bmatrix} = \frac{1}{\chi}O(\eta^2); \tag{8·9a}$$

by using the restriction (7·20*c*) on χ, we get also

$$\begin{bmatrix} A' \\ \omega \\ WA' \end{bmatrix} = O(\eta^2) \quad \text{and} \quad \begin{bmatrix} B' \\ \omega \\ WB' \end{bmatrix} = O(\eta^2). \tag{8·9b}$$

It is convenient to rewrite (8·1) in another form. If we substitute (8·3), (8·6) and (8·9) into (8·1), we get

$$[a] = \frac{1}{\chi}(P+p)\,[b], \tag{8·10a}$$

$$[b] = \frac{1}{\chi}(Q+q)\,[a], \tag{8·10b}$$

where P and Q are the matrices defined in (8·3), and

$$p \equiv \begin{bmatrix} 0 & 0 \\ p_0 & 0 \end{bmatrix} + O(\zeta^5) + O(\eta^2), \tag{8·11a}$$

$$q \equiv \begin{bmatrix} 0 & 0 \\ q_0 & 0 \end{bmatrix} + O(\zeta^5) + O(\eta^2). \tag{8·11b}$$

To throw (8·1) into the form (8·10), it was necessary to take the terms

$$\left(\begin{bmatrix} A' \\ \omega \\ RA' \end{bmatrix} + \begin{bmatrix} A' \\ \omega \\ WA' \end{bmatrix}\right)[a] \quad \text{and} \quad \left(\begin{bmatrix} B' \\ \omega \\ RB' \end{bmatrix} + \begin{bmatrix} B' \\ \omega \\ WB' \end{bmatrix}\right)[b]$$

in (8·1) over to the left-hand side, and multiply through by the inverse of the resulting matrix on the left. (According to (8·6) and (8·9b), it is possible to choose η_1 and ζ_1 so that the inverse exists when $\eta < \eta_1$ and $\zeta < \zeta_1$.) The new terms appearing on the right as a result of the multiplication are absorbed in the O terms in (8·11).

The condition for our model to be able to act as a dynamo is that the equations (8·10) should have a solution. We next consider the conditions for this to be so. By substituting (8·10b) into (8·10a), we get

$$\left[I - \frac{1}{\chi^2}(P+p)(Q+q)\right][a] = 0, \tag{8·12}$$

where I is the unit matrix. Equations (8·10) will have a non-trivial solution if (8·12) has, and the condition for this is that

$$\det[\chi^2 I - (P+p)(Q+q)] = 0. \tag{8·13}$$

The matrix $(P+p)(Q+q)$ is found from (8·3) and (8·11) to be

$$(P+p)(Q+q) = \begin{bmatrix} \chi'^2 + P_{12}q_0 + O(\zeta^5) & O(\zeta^5) \\ p_0 Q_{11} + q_0 P_{22} + O(\zeta^5) & \chi'^2 + Q_{12}p_0 + O(\zeta^5) \end{bmatrix}, \tag{8·14}$$

where $$\chi'^2 \equiv \tfrac{1}{3}(\sin\Theta_A \sin\Theta_B \sin\Lambda_{AB})^2 [\cos\Theta_A \cos\Theta_B - \sin\Theta_A \sin\Theta_B \cos\Lambda_{AB}], \tag{8·15}$$

and where P_{ij} and Q_{ij} are the elements of P and Q. In writing (8·14), we have put

$$\eta = O(\zeta^{\frac{5}{2}}). \tag{8·16}$$

This has the effect of incorporating all terms in (8·12) depending explicitly on η in the $O(\zeta^5)$ terms in (8·14); since these are smaller than the terms containing p_0 and q_0 by a factor $O(\zeta^2)$, (8·16) means that we are choosing the parameters so as to make the interaction with the conductor surface predominant among the small effects. With the aid of (8·14), the condition (8·13) for dynamo action becomes

$$[\chi^2 - \chi'^2 - P_{12}q_0 + O(\zeta^5)][\chi^2 - \chi'^2 - Q_{12}p_0 + O(\zeta^5)] + O(\zeta^8) = 0. \tag{8·17}$$

Before discussing this equation, we shall prove that the left-hand side is a real and continuous function of ω when ω is real. χ^2 is manifestly real and continuous (see (8·4)). The other terms in (8·17) are also real and continuous because the matrices in (8·1) are. That these matrices are real can be seen as follows: a real electromagnetic field always produces real fields by induction in a rotator or reflexion at the conductor surface (if χ is real). Now if $\bar{a}_{T10}$, a_{T20}, $\bar{b}_{T10}$ and b_{T20} are real, then the fields $\bar{a}_{T10}(A'T10)$, $a_{T20}(A'T20)$, $\bar{b}_{T10}(B'T10)$

and $b_{T20}(B'T20)$ are also real, so that the part of the electromagnetic field which is described by the weak a's and b's is real, and can induce only real contributions to $\bar{a}_{T10}$, a_{T20}, $\bar{b}_{T10}$ and b_{T20}. It follows that the matrices in (8·1) containing W are real. The other matrices in (8·1) can also be shown to be real, either by similar arguments, or by inspection. The continuity of the matrices P, Q, p and q is proved by noting that those matrices in (8·1) which do not contain W are continuous functions of χ, which they contain only as a factor χ^{-1}. Those matrices which do contain W are also continuous because the series (8·7) which defines them is uniformly convergent with respect to ω—for our method of computing the upper bound (8·8) shows that Weierstrass' M-test can be satisfied—and because of the continuity of the weak a's and b's which was proved at the end of § 7.

We can now discuss the dynamo condition (8·17). Let us see first what happens if we ignore the detailed structure of the terms containing ζ, and rewrite (8·17) in the form

$$(\chi^2-\chi'^2)^2+O(\zeta^3) = 0. \tag{8·18}$$

Since $\chi^2>0$ when ω is real, it follows that a necessary condition for (8·18) to have a solution for a real value of ω is

$$\chi'^2>0 \tag{8·19a}$$

if ζ is small. This condition is, however, not sufficient because the term $(\chi^2-\chi'^2)^2$ in (8·18) is positive definite, so that it becomes essential to consider the detailed structure of the O term. This means that terms in (8·1) which become vanishingly small as $\eta\to 0$ and $\zeta\to 0$ are, nevertheless, of decisive importance for whether dynamo action is possible or not. We therefore return to the dynamo condition as written in (8·17). The term [...] [...] will vary approximately parabolically with χ^2 if ζ is small, and will have two distinct zeros if

$$P_{12}q_0-Q_{12}p_0 \neq 0,$$

or

$$-\sin\Theta_A\sin\Theta_B\sin\Lambda_{AB}(\sin\Theta_A\sin\Theta_B\cos\Lambda_{AB}-4\cos\Theta_A\cos\Theta_B)$$
$$\times\frac{1}{24}\frac{R^2}{\bar{R}^3}\zeta^3(\mathbf{R}_A+\mathbf{R}_B).(\hat{\boldsymbol{\omega}}_A\wedge\hat{\boldsymbol{\omega}}_B)\neq 0. \tag{8·19b}$$

When χ^2 lies between the two zeros, the term [...] [...] is negative and its magnitude reaches a maximum value of $\propto\zeta^6$; when χ^2 lies outside the two zeros, the term [...] [...] is positive and unbounded. Since everything in (8·17) is real and continuous, and since the second term in (8·17) is $O(\zeta^8)$, it follows that when ζ is small, equation (8·17) has two roots

$$\chi^2 = \chi'^2+O(\zeta^3); \tag{8·20}$$

it also follows that the conditions (8·19) are sufficient for the strong equations to have a solution.

We should note that ζ (and also η) has to be small, not only to satisfy (8·17) but also to satisfy the conditions (7·12a), (7·34), and $\eta<\eta_1$, $\zeta<\zeta_1$ (after (8·11)). Our calculations are not precise enough to give upper bounds below which η and ζ must lie; however, we know that such upper bounds exist and are non-zero, so that it is *not* necessary to go to the limit $\eta\to 0$, $\zeta\to 0$ for our conclusion that our model can act as a dynamo to be valid.

Since the discussion leading up to (8·20) made use of the results of § 7, it is necessary that $|\chi|$ should exceed an arbitrarily chosen non-zero lower bound χ_0 (see (7·20b)). This can be arranged by choosing $0<\chi_0<|\chi'|$.

We should note that since $\omega \equiv |\boldsymbol{\omega}_A| = |\boldsymbol{\omega}_B|$, χ is positive by definition, and therefore only the positive value χ obtained from (8·20) has any significance.

The demonstration that the strong equations can have a solution completes our proof that steady dynamos are possible in principle.

9. The magnetic field outside the conductor

To be of interest, a dynamo mechanism must not only be able to maintain its electromagnetic field, but must also produce a magnetic field outside the conducting material. In this section, we shall prove that our mechanism can do this.

The part of the magnetic field in $r > M$ (in terms of figure 1) which falls off least rapidly with r is the dipole component which decreases as r^{-3}. It will be sufficient to show that this component is non-zero. This can be done by calculating the external magnetic field arising from $(A') + (B')$ in the way described at the end of § 5.

The only components of (A') and (B') which contribute to the dipole field are $(A' \text{ or } B'\ S1m)$ and $(A' \text{ or } B'\ T1m)$. (The reason why a toroidal field with $n = 1$ makes a contribution varying as r^{-3} and not as r^{-2} is its transverse nature, since

$$(A'\,Tnm;\,\mathbf{H}) = \sigma \boldsymbol{\nabla} \alpha'_{nm} \wedge \mathbf{r}_A;$$

the vector $(A'\,Tnm;\,\mathbf{H})$ has a magnitude proportional to r^{-n-1}, but its projection on $\hat{\mathbf{r}}$, which determines the magnitude of the contribution outside the conductor, falls off more rapidly by another factor r.) From our equations (7·35) and (7·26), we see that the coefficients of the field $(A' \text{ or } B'\ S \text{ or } T\ 1m)$ with $m \neq 0$ are smaller than those of the fields with $m = 0$ by several powers of η; we need, therefore, consider only the fields $(A'\,T10)$ and $(B'\,T10)$. To avoid a discussion of the interference of these two fields, we shall suppose that the rotator B is concentric with the conducting shell. The discussion of § 5 then shows that $(B'\,T10)$ gives no magnetic field in $r > M$; we shall therefore have proved that our dynamo can produce an external magnetic field if we can show that the coefficient a_{T10} of $(A'\,T10)$ is not zero.

If we put $\chi^2 = \chi'^2 + q_0 P_{12} + O(\zeta^5)$ in (8·12), and use (8·14), then we find that

$$\frac{\bar{a}_{T10}}{a_{T20}} = \frac{P_{12} q_0 - Q_{12} p_0 + O(\zeta^5)}{P_{22} q_0 + Q_{11} p_0 + O(\zeta^5)}. \tag{9·1}$$

The numerator of this expression, and therefore $\bar{a}_{T10}$, is non-zero if the condition (8·19*b*) is satisfied. This means that the conditions which make a dynamo possible are also sufficient to give a magnetic field outside the conductor.

10. Physical significance of the model

A complete discussion of the physical significance of our dynamo model is clearly impossible because the velocity distribution of the conducting fluid has been postulated, instead of being derived from the equations of motion. However, it does seem worth while to discuss the following two questions:

(1) How probable or improbable are the relative orientations of the axes of the two rotators which give dynamo action?

(2) How do the velocities required by the model for dynamo action compare with our empirical knowledge of velocities in the earth's core?

To answer the first question, we recall from § 8 ((8·15) and (8·19a)) that when η and ζ are small, the condition for dynamo action to be possible is

$$\cos\Theta_A \cos\Theta_B - \sin\Theta_A \sin\Theta_B \cos\Lambda_{AB} > 0. \tag{10·1}$$

(For the moment, we shall ignore the condition (8·19b) and the possibility of the vanishing of the factor $(\ldots)^2$ in (8·15).) It is easily shown that if (10·1) is not satisfied for a particular relative orientation of $\hat{\boldsymbol{\omega}}_A$ and $\hat{\boldsymbol{\omega}}_B$, then it will be satisfied if either one of $\hat{\boldsymbol{\omega}}_A$ or $\hat{\boldsymbol{\omega}}_B$ is reversed. Conversely, if (10·1) is satisfied, then a reversal of either $\hat{\boldsymbol{\omega}}_A$ or $\hat{\boldsymbol{\omega}}_B$ will violate it. The proof of these statements is that the left-hand side of (10·1) changes sign if we replace (Θ_A, Λ_A) by $[(\pi-\Theta_A), (\pi+\Lambda_A)]$ or (Θ_B, Λ_B) by $[(\pi-\Theta_B), (\pi+\Lambda_B)]$; (remember that $\Lambda_{AB} = \Lambda_A - \Lambda_B$). It follows that precisely half the possible relative orientations of $\hat{\boldsymbol{\omega}}_A$ and $\hat{\boldsymbol{\omega}}_B$ lead to dynamo action if $|\boldsymbol{\omega}_A|$ and $|\boldsymbol{\omega}_B|$ are suitably adjusted. This conclusion is not affected by the condition (8·19b) and the condition that the factor $(\ldots)^2$ in (8·15) should not vanish, requirements which we have ignored up till now. The reason for this is that the relative orientations of $\hat{\boldsymbol{\omega}}_A$ and $\hat{\boldsymbol{\omega}}_B$ which violate these conditions form only a very small fraction of all those possible.

We should add that the condition (10·1) on which the discussion of the preceding paragraph was based is not exact because the terms $P_{12}q_0$ and $Q_{12}p_0$ in (8·17) were omitted. If these terms are taken into account, then it remains true to say that roughly one-half of the possible relative orientations constitute dynamos if η and ζ are small.

We next consider the velocities required by our model for dynamo action. The peripheral velocity ωa of the rotators can be found from (8·4), (8·15) and (8·20). To get a typical value, we shall take $\Theta_A = \Theta_B = \frac{1}{4}\pi$ and $\Lambda_{AB} = \frac{1}{2}\pi$. One then finds that

$$\omega a = 40\sqrt{24}\,\frac{1}{\eta^3 a\sigma}, \tag{10·2}$$

when η and ζ are small. To get an estimate of some practical significance, we shall ignore the restrictions on η and ζ, and apply (10·2) to a case in which our model roughly speaking fills out the core of the earth. The estimate for the velocities which we shall obtain is that corresponding to a calculation with neglect of all small effects, i.e. taking into account only the large terms in the strong equations. We put

$$\eta = \tfrac{1}{2}, \quad a = \tfrac{1}{3}a_c,$$

where a_c is the radius of the earth's core. We then get from (10·2)

$$\omega a = \frac{4730}{\sigma a_c}. \tag{10·3}$$

This estimate is larger by a factor 25 than one obtained in a numerical treatment of the problem of the dynamo in the earth's core by Bullard & Gellmann (1954), who give $v_r(\text{max.}) = 187/\sigma a_c$ for the maximum value of the radial component of velocity. (See equation (58) on p. 271 of their paper. Note that Bullard & Gellmann use unrationalized e.m.u., so that we have to put $4\pi\kappa \rightarrow \sigma$ in their formula, that their a is our a_c, and their

$V \simeq 70$.) It is worth noting that the velocity pattern treated by Bullard & Gellmann is quite different from that of our model.

If one takes $\sigma = 3\times10^{-5}$ e.m.u. (Bullard & Gellmann 1954) and $a_c = 3\times10^8$ cm for the radius of the earth's core, then equation (10·3) gives $\omega a = 0{\cdot}5\,\mathrm{cm\,sec^{-1}}$. This is larger by about an order of magnitude than the velocity at which the non-dipole field drifts westward with respect to the earth's mantle, this being the only velocity in the earth's core which can be measured at present.

11. Discussion

(a) *Conditions for dynamo action*

We have shown that our two-rotator model drawn in figure 1 can act as a dynamo if η and ζ are small, and can give rise to a magnetic field outside the conductor (see §9); $(\eta \equiv 2a/R,\ \zeta \equiv 2(|\mathbf{R}_A|+|\mathbf{R}_B|)/M)$. The only important conditions for dynamo action to be possible were that the angular velocities should be high enough (see (8·4), (8·20), (10·3)), and that the orientations of the axes of rotation should satisfy (10·1), which is satisfied by about half of the possible relative orientations.

It should be emphasized that dynamo action has been proved to be possible for finite values of η and ζ, and that it is *not* necessary to go to the limit $\eta = 0$ and $\zeta = 0$ for our proof to be valid. However, our calculations are not precise enough to give upper bounds below which η and ζ must lie if dynamo action is to be possible with certainty. What happens when η and ζ are not small, we cannot say; however, there is no obvious reason to suppose that our model cannot act as a dynamo when η is close to 1, and ζ close to 4, the upper limits permitted by the geometry. The requirement that η and ζ be small is merely a convenient mathematical device for making the equations tractable.

(b) *Physical significance of the model, and the relation to the model of Bullard & Gellmann*

Our model cannot have any direct application to movements in the core of the earth because it makes no attempt to satisfy the equations of motion. All it does is to provide an existence theorem that steady dynamos in a conducting sphere are possible in principle. Nevertheless, it is interesting that the velocities we have estimated with our dynamo differ by no more than an order of magnitude from those obtained by Bullard & Gellmann (1954) (B.–G.) for quite a different system of motions which might well be a fairly good representation of what happens in the earth's core; their velocity estimate agrees in order of magnitude with what is observed in the westward drift (Bullard *et al.* 1950).

It is interesting to compare in some detail the model discussed in this paper with that treated by B.–G. They too discussed motions in a sphere of conducting fluid. The velocity pattern of their model is shown in figure 11 of their paper; it consists, roughly speaking, of four eddies whose centres lie on the equator, and whose angular velocity vectors point alternately from north to south and south to north. Superimposed on these eddies, there is a non-uniform rotation of the whole fluid about the north-south axis. Compared with the model described in this paper, that of B.–G. has one advantage and one disadvantage. The advantage is that the B.–G. model probably bears a much better resemblance to what goes on in the core of the earth than does our model. The disadvantage of the B.–G. model is that it was not possible to give a mathematically satisfactory demonstration of its ability to act as a dynamo, whereas the proof given for our model is rigorous.

The model described in this paper differs from that treated by B.–G. on three points:

(1) The non-uniform rotation of B.–G. has no counterpart in our model.

(2) B.–G. use four eddies, whereas we have used only two.

(3) The mathematical representation of the fields and velocities are different.

We next consider these points in detail.

Point (1)

Before the work of B.–G., the non-uniform rotation was thought to be an essential component of a dynamo consisting of motions in a fluid sphere. This hypothesis was not borne out by their results. According to table 4 of their paper, a gradual increase of the velocity in the non-uniform rotation from zero to one hundred times that in the eddies reduces the critical eddy velocities by less than a factor 2. Moreover, not only does the non-uniform rotation fail to be of any significant help in the working of their dynamo, it can actually be a hindrance, because it is apparently possible to show that if the ratio of its velocity to that of the eddies tends to infinity, then no dynamo is possible (Bullard, private communication).

These remarks should not be taken to imply that the non-uniform rotation is of no physical importance. There are good reasons for supposing that it exists, and moreover, it may well be essential to explain the westward drift of the non-dipole field (Bullard *et al.* 1950). But in the basic process of self-excitation of the B.–G. dynamo, the non-uniform rotation plays no important role. The fact that it is present in the B.–G. model and absent from ours is therefore of no great significance.

Point (2)

Since the B.–G. dynamo has been shown, within their approximations, to work without the non-uniform rotation of the whole fluid sphere, i.e. with the eddies alone, the essential physical difference between their model and ours is in the number of eddies. That the characteristic velocity obtained in § 10 of this paper is so much larger than that of B.–G. is probably largely due to our eddies being somewhat smaller than theirs. A point to note is that the B.–G. 'eddies' are not simply rigid body rotations like ours, but contain components of velocities perpendicular to their equatorial planes; however, it is likely that much the same effect could be achieved by four rigid eddies whose axes are slightly tilted out of the north-south direction.

Point (3)

The difference between the mathematical representations of the velocities and fields in this paper and that of B.–G. is responsible for the fact that our treatment is rigorous, whereas B.–G. had to use infinite series cut off without justification. Our representation is the more flexible of the two; it allows us to vary freely and independently the radii of the eddies and their position in the large sphere of conducting fluid. It was just these two degrees of freedom (represented by the parameters η and ζ) which lay at the base of our treatment. There was no corresponding freedom in the treatment of B.–G.

(*c*) *Undesirable physical features of the model*

Our model contains some features which are physically improbable; these are the requirements that the two rotators should have the same angular velocity and radius, and that they should have sharp surfaces.

The requirements of equal angular velocities and radii are of no consequence, because a re-examination of the argument of § 8 shows that if the two rotators were given different radii a and b, and different angular velocities ω_A and ω_B (we may assume $\omega_A\omega_B > 0$ without loss of generality), then all that happens is that χ^2 in (8·12) has to be redefined to be

$$\chi^2 \equiv [(\omega_A a^2 \sigma)(\omega_B b^2 \sigma)(2a/R)^3 (2b/R)^3/40^2]^{-1},$$

the definition of P, Q, p and q remaining unchanged. It follows that our conclusions about the conditions for dynamo action and the possibility of generating a field outside the conductor remain unaffected; the result (10·3) continues to be a reasonable estimate of the velocities required.

The sharp surfaces of the rotators are more serious. To speculate on the possible effect of a diffuse surface, let us recall that the properties of rotating conductors in magnetic fields entered into the argument only in two essential ways: we have used, in the first place, the ability of such a rotator to produce an induced magnetic field of unlimited magnitude from an applied magnetic field with axial symmetry, and, in the second place, the saturation effect which limits the induced field when the applied field does not have axial symmetry (see § 4). A diffuse edge is unlikely to affect the ability to produce an unlimited field from an axially symmetric one (Herzenberg & Lowes 1957). However, the effect of a diffuse edge on the saturation effect for a transverse applied field is an open question (it was shown in the paper just cited that the induced field near an *accelerating* rotator in a magnetic field perpendicular to the axis can be seriously affected by a diffuse edge).

(*d*) *Essential features of the mechanism*

Three different physical processes enter into the working of our model.

(1) The direct interaction of the two rotators through the magnetic fields associated with what we have called the strong coefficients a and b. (By direct interaction we mean one that does not involve the conductor surface.)

(2) The direct interaction through the weak a's and b's.

(3) The interaction with the surface of the conductor.

We have chosen the parameters so that process (1) should be predominant and provide most of the interaction; the physical meaning of the most important condition for dynamo action (i.e. (8·19*a*)) is just that the feedback due to this mechanism should be regenerative.

Processes (2) and (3) contribute little to the magnitude of the interaction, but are, nevertheless, very important because they enter into the second condition (8·19*b*) for dynamo action in an essential way and not only as small correction terms. This comes as a surprise, somewhat mitigated by the fact that if one chooses, as we have done, the parameters η and ζ so that surface interaction predominates over coupling through the 'weak' fields, then the second dynamo condition is almost automatically satisfied. It is impossible to say whether the importance of the small effects is a peculiar feature of our model, or more general. Nevertheless, its importance in our case does suggest that it might be dangerous to treat the dynamo problem with series which are cut off arbitrarily.

Our choice of the parameters so that process (3) should predominate over process (2) was somewhat arbitrary, and by making (ζ/η) small we could have had it the other way. The reason for our choice was mathematical simplicity. Until the alternative choice has been worked out, it will be impossible to say whether the boundedness of the conductor in which the motions take place is an essential feature of dynamos in a continuous fluid, or whether dynamos are also possible when the influence of the conductor surface is negligible.

(e) *Possibility of a dynamo with a single rotator*

The importance of surface interaction in our model suggests the possibility of a dynamo consisting of a single rotator in a bounded conductor, the feedback being due to reflexion from the surface. If this were possible, then a single convection cell in the core of the earth might be able to act as a dynamo. It should be possible to treat this problem by methods similar to those used here.

(The case of a spherical rotator *concentrically* placed within a spherical shell has been treated by Bullard (1949) and found not to give dynamo action. However, this case is rather special because one can show from the results in § 5 that the field reflected from the boundary cannot have the form $(ASn0)$ necessary for amplification by the rotator; it would certainly be necessary to place the rotator excentrically.)

(f) *The appearance of the magnetic field*

It is not difficult to form a picture of the field when $\eta \ll 1$ and $\zeta \ll 1$. The induced field at each rotator is then much stronger than the applied field, so that one gets two islands of strong field at the rotators, and very little in between. Moreover, the only field components which are important in the islands will be $(A'T20)$ and $(B'T20)$ (because the coefficients are of the same order (see § 8 and (7·26)). Therefore in each island, the magnetic field is nearly axially symmetric, and points along parallels of latitude in opposite directions on opposite sides of the equator of the rotator belonging to the island.

12. Conclusion

It is possible for a model consisting of two steadily rotating solid conducting spheres embedded in a region of solid conducting material bounded by a sphere to act as a dynamo producing a magnetic field outside the conductor. About half of the possible configurations can act as dynamos if (i) the velocities are suitably adjusted, (ii) the radii of the rotating spheres are small compared with the distance between their centres, and (iii) the distances of the rotating spheres from the centre of the conductor in which they are embedded are small compared with its radius.

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